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BY

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PREFACE

The author is a Line Officer of the Regular Navy with the present rank of Commander and has pursued his graduate studies at the U.S. Naval Postgraduate School, Monterey, California and Indiana University. I desire to acknowledge the sponsorship in these studies of the Chief of Naval Personnel, the Chief of Naval Research and the Superintendent, U.S. Naval Postgraduate School.

I became interested in the theory of non-Newtonian fluids during a series of brilliant and inspiring lectures by Professor C. Truesdell at Indiana University in 1951-52 during which many of the problems solved herein suggested themselves. I served at sea during most of the intervening years and postponed many times the solutions of these problems.

I wish to thank specifically Professors D. Gilbarg, formerly of Indiana, now of Stanford University; C. Truesdell; and J.W.T. Youngs, Chairman of the Mathematics Department of Indiana University for their encouragement, inspiration and patience both at the University and through the mails over the years; J.L. Ericksen, Johns Hopkins University, for suggestions and review of the draft of this paper; and Dr. J. Weyl, Science Director of the Office of Naval Research, who has advised me wisely since my first association with him in 1950.

I cannot complete this preface without recording acknowledgement of my beloved wife, Dailey, who has for the past year, and from time to time before, coped uncomplainingly with a spouse bound to two most exacting tasks, my professional duties on the Staff of the Chief of Naval Operations in Washington and the completion of this dissertation.

CONTENTS

Chapter		Page
I.	OBJECTIVES AND SUMMARY	1
II.	SYMBOLS, DEFINITIONS AND EQUATIONS	4
III.	FLUID DEFINITIONS	9
IV.	GENERAL EFFECTS OF NON-LINEAR THEORY	14
V.	SIMPLE POISEUILLE FLOW	17
VI.	POISEUILLE FLOW IN A CIRCULAR PIPE	24
VII.	STRUCTURE OF SHOCK WAVES	29
VIII.	STEADY FLOW BETWEEN NON-PARALLEL WALLS	41
IX.	THE PROBLEM OF BOUSSINESQ	50
X.	THE DRAINING OF A VERTICAL PLATE	59
	BIBLIOGRAPHY	67
	APPENDIX Derivation of the equations of Hydrodynamics	71

I. OBJECTIVES AND SUMMARY

This dissertation treats the examination of the effect on fluid flow of the theory of non-Newtonian fluids. The general methods of Reiner and Rivlin, as developed by Truesdell, are followed.

An investigation is made in Chapter 4. of the functional dependence of the pressure on the space variables and its effect on the possibility of existence of a particular flow. This functional dependence is closely related to the form of the second order viscosity coefficients. In many cases, making the same assumption of such dependency of pressure as is made in the linear case, together with taking the viscosity coefficients as constants, forces only the linear solution. This may lead to the erroneous conclusion that the particular flow of the non-Newtonian fluid is not possible. This functional dependence is discussed in the general case in Chapter 4. and in many of the particular flows studied.

The stress tensor is developed in such a manner as to emphasize the Kelvin and Poynting effects of the non-Newtonian theory.

Several specific flows are considered. These flows were chosen as a result of many considerations, the primary one of which was, of course, that new results could be obtained. Other considerations prompting the selection of specific flows

were: (1) the form of the rate of deformation tensor, (2) the coordinate system in which the flow could be considered, (3) variation of observable phenomenon, and (4) solvability of the equations of motion.

Chapter 2. reviews the symbols, definitions and equations which are further developed in the Appendix. The importance of complete understanding of physical components of tensors and their relationships to the other tensor components cannot be overemphasized when computations are necessary.

Chapter 3. reviews the definitions of fluids and develops the general form of the extra stress tensor which is used primarily herein for the explicit display of the terms contributing to the Kelvin and Poynting effects.

Chapter 4. discusses the latter effects and the general effect of the pressure on the non-linear theory.

In Chapter 5., simple Poiseuille flow is studied. By means of inversion of series, a general solution is provided. An explicit form of the pressure is also produced. It is shown that the pressure is dependent on the downstream variable, a phenomenon not observed in the linear case.

Poiseuille flow of a non-Newtonian fluid in a circular pipe is completely solved in Chapter 6. In addition, a generalization of the Hagen-Poiseuille Efflux Law is given.

In Chapter 7., we demonstrate the existence and study the structure of shock waves in non-linear continuum theory. The steady one-dimensional flow of a trivariate, heat conducting fluid with non-linear viscosity is considered in the absence of external forces. The discussion is limited to perfect gases. However, the proof of existence carries over to fluids with more general equations of state. Geometrical properties of solutions to systems of differential equations are employed to show existence. Comparison is made of thicknessess of shock waves in non-linear fluids and linear fluids after choosing an arbitrary form of the viscous stress.

The steady flow of a non-Newtonian fluid in a wedge or diverging channel is studied in Chapter 8. This flow provides an excellent example of the computational methods necessary in the non-linear theory. We are lead to the same elliptic integral solution as in the classical case. The pressure distribution is, of course, different.

The problem of Boussinesq is considered in several general cases. The effects of pressure dependence are very noticeable in this problem. Consideration of the dependency of pressure on the distance from the boundary is mandatory.

The equations of motion of the draining of a vertical plate are solved in Chapter 10. Quantitative information is provided and a set of curves showing comparisons of thicknesses of the linear and non-linear theories with observed results are provided.

II. SYMBOLS, DEFINITIONS AND EQUATIONS

General. - Insofar as is possible, standard symbols of modern fluid dynamics will be used throughout the text. The symbols and conventions of tensor analysis[1] are used wherever general terms or equations are written. Each symbol is defined when first introduced.

In actual computations, physical components are used. The beauty and elegance of the invariant form of equations unfortunately do not generally persist when the physical components are substituted for the flow variables. On the other hand, physical interpretation of tensor components is not always clear. The equations using physical components and the comparable ones employing the invariant forms can indeed be quite different in any but a rectangular cartesian coordinate system. In fact, it is this difference that lead Truesdell[2] to write "in 1944, I first had the puzzling experience of getting an apparently wrong form of the equations of elasticity in polar coordinates by specializing the perfectly correct general tensor equation to the case at hand."

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[1] C. E. Weatherburn, Riemannian Geometry and the Tensor Calculus. (Cambridge: University Press, 1938).

[2] C. Truesdell, "The physical components of vectors and tensors," Zeitschrift für angewandte Mathematik und Mechanik, XXXIII, (1953), 345-356.

We use $t(ij)$ to mean the physical components of the tensor t^i_j.

We assume, of course, that a fluid is a continuous medium, with which the reader is presumed to be familiar. Equations are derived in the Appendix, but are given below.

Equation of Continuity.

Let ρ be a fluid density

$\dot{x}^i$ be the contravariant components of velocity where $\dot{}$ is the symbol for material differentiation,

then the equation of continuity takes the form,

$$\frac{\partial \rho}{\partial t} + (\rho \dot{x}^i)_{,i} = 0 \tag{2-1}$$

where " $,i$ " is the symbol for covariant differentiation.

Cauchy's Laws of Motion.

Let t^{ij} be the stress tensor (symmetric)[3]

f^i be an extraneous force field, and,

$\ddot{x}^i$ the components of acceleration,

then,

$$t^{ij}{}_{,j} + \rho f^i = \rho \ddot{x}^i \tag{2-2}$$

is Cauchy's first law of motion.

[3]See Appendix for definition of t^{ij}.

Energy Equation.

Let ε be the internal energy density, and

q^i be the heat flux vector,

then

$$\rho\dot{\varepsilon} = t^{ij}\dot{x}_{i,j} - q^i{}_{,i}.$$

Defining the rate of deformation tensor as usual,

$$d_{ij} \equiv \frac{1}{2}(\dot{x}_{i,j} + \dot{x}_{j,i})$$

and recalling the symmetry of t^{ij} we obtain

$$\rho\dot{\varepsilon} = t^{ij} d_{ij} - q^i{}_{,i} \qquad (2\text{-}3)$$

the Fourier-Kirchoff-C. Neumann equation.

Extra Stress Tensor.

Defining the extra stress tensor, v^i_j , by

$$v^i_j \equiv p\,\delta^i_j + t^i_j \qquad (2\text{-}4)$$

where δ^i_j is the Kronecker delta, and p is any scalar. Then the equation of energy can be placed in the form,

$$\rho\,\theta\,\dot{\eta} = (\pi - p)\frac{\dot{v}}{v} + v^{ij} d_{ij} - q^i{}_{,i}$$

where η is a local statistical parameter (entropy), and

$$v \equiv \frac{1}{\rho} \qquad \pi \equiv -\frac{\partial\varepsilon}{\partial v} \qquad \theta \equiv \frac{\partial\varepsilon}{\partial\eta}.$$

For incompressible flow ($\dot{\rho} = 0$), $\dot{\mathcal{V}} = 0$. In order that we may obtain the same elegant result in the compressible case, we choose the scalar $p = \pi$[4] and obtain,

$$\rho \theta \dot{\eta} = v^{ij} d_{ij} - q^i{}_{,i}$$

a different form of the energy equation. In the flows considered herein, we assume that the stress power is non-negative, that is

$$t^{ij} d_{ij} \geqslant 0$$

or equivalently,

$$v^{ij} d_{ij} \geqslant 0 .$$

Once we assume an equation of state, say

$$\theta = \Theta(\rho, \eta)$$

a given form,

$$q_i = -\kappa \, \theta_{,i} \qquad \text{(Fourier's Law)}$$

and the form of the extra stress tensor

$$v^i_j = f(d^i_j, \theta, p, a, b, c, \ldots)$$

where f[5] denotes functional dependence and a,b,c.... are constants of the fluid, then we have all the equations necessary to study the flow of fluids considered herein and many others.

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[4] C. Truesdell, "On the Viscosity of Fluids According to the Kinetic Theory," Zeitschrift für Physik, CXXXI, (1952), 273-289.

[5] f is used throughout to denote functional dependence rather than a specific function.

Classification of Fluids and Motions. - If $\dot{\rho} = 0$ in all motions, then the fluid is said to be incompressible. Furthermore, if ρ is an absolute constant, the fluid is said to be homogeneous and incompressible.

If $p = p(\rho)$ in all motions, then the fluid is said to be piezotropic.

If $p = p(\rho, \eta)$ in all motions, then the fluid is said to be trivariate.

If $\rho = \rho(t)$, the motion is said to be isochoric; if $p = p(t)$, isobaric; if $\rho = \rho(p, t)$ or $p = p(\rho, t)$, barotropic. All other motions are said to be baroclinic.

III. FLUID DEFINITIONS

The general form of the extra stress tensor. - The definition of the extra stress tensor is a mathematical formulation of Stokes observation[1] "that the difference between the pressure on a plane in a given direction (i.e., a component of stress) passing through a point P of a fluid in motion, and the pressure which would exist in all directions at P (i.e., the hydrodynamic pressure), if the fluid were in a state of relative equilibrium, depends only on the relative motion of the fluid immediately about P, and that the relative motion due to motion of rotation may be eliminated without effecting the difference of the pressure above mentioned." Thus, following Reiner[2], Rivlin[3,4], and Truesdell[5], we write

$$t^i_j + p\,\delta^i_j = \mathcal{V}^i_j \tag{3-1}$$

where $\mathcal{V}^i_j = 0$, if $d^i_j = 0$.

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[1] G. G. Stokes, "On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids," Transactions of the Cambridge Philosophical Society, VIII, (1845), 287-319.

[2] M. Reiner, "A mathematical theory of dilatancy," American Journal of Mathematics, LXVII, (1945), 350-362.

[3] R. S. Rivlin, "Hydrodynamics of non-Newtonian fluids," Nature, CLX, (1947), 611-613.

[4] R. S. Rivlin, "The hydrodynamics of non-Newtonian Fluids I," Proceedings of the Royal Society of London, CXCIII, (1948), 260-281.

[5] C. Truesdell, "A new definition of a fluid, I. The Stokesian fluid," Journal de Mathematiques Pures et Appliquées, XXIX, (1950), 215-244.

We consider only isotropic[6] non-Maxwellian fluids, and set

$$v^i_j = A\left[G_0 \delta^i_j + BG_1 d^i_j + B^2 G_2 d^i_k d^k_j\right] \tag{3-2}$$

where A, B and G_γ will be defined below. For ease in identifying Newtonian and non-Newtonian fluids and the higher order viscosity effects, we re-write (3-2) in a more convenient form,

$$\begin{aligned} v^i_j = {} & \lambda d^k_k \delta^i_j + 2\mu d^i_j \\ & + K_0 \delta^i_j + K_1 d^i_j + K_2 d^i_k d^k_j \end{aligned} \tag{3-3}$$

where $\lambda + \frac{2}{3}\mu$ is the bulk viscosity[7], μ is the shear viscosity and

$$\begin{aligned} K_0 &= AG_0 - \lambda d^k_k \\ K_1 &= ABG_1 - 2\mu \\ K_2 &= AB^2G_2 . \end{aligned} \tag{3-4}$$

(3-3) is an elegant form of the extra stress tensor from which we shall take departure in defining all non-Maxwellian fluids. From (3-3), the higher order viscosity effects become immediately evident.

Newtonian Fluid.

If all $K_\gamma = 0$, we have from (3-3) the Newton-Cauchy-Poisson form of the extra stress tensor,

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[6] Ibid. 223-224.

[7] C. Truesdell, "The present status of the controversy regarding the bulk viscosity of fluids," Proceedings of the Royal Society, CCXXVI, (1954), 59-65.

$$v^i_j = \lambda d^k_k \delta^i_j + 2\mu d^i_j$$

which is the characteristic equation of the classical theory of fluids. All results of the theory of linear viscosity, of course, stem from this relationship.

Stokesian Fluid. - Truesdell[8] characterized the Stokesian fluid as a fluid without natural time, i.e., a body which offers no response to its previous form or a "body without a memory."

If we take

$$A = p$$
$$B = \frac{\mu_n}{p}$$

where p, for incompressible fluids is an unspecified scalar, while for compressible fluids, p is the thermodynamic pressure, and where μ_n is the natural viscosity (dimension $\frac{M}{LT}$), then (3-4) gives

$$K_0 = p G_0 - \lambda d^k_k$$
$$K_1 = \mu_n G_1 - 2\mu$$
$$K_2 = \frac{\mu_n^2}{p} G_2$$

which characterize the Stokesian fluid.

Reiner - Rivlin fluid. - We continue to follow Truesdell[9]

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[8] Truesdell, "A new definition of a fluid, I.," XXIX

[9] Ibid.

by adding a third material constant t_n (dim T), obtaining a fluid with a natural time. The K_r characteristics of the Reiner-Rivlin fluid are obtained by taking

$$A = \frac{\mu_n}{t_n}$$
$$B = t_n$$

whence,

$$K_0 = \frac{\mu_n}{t_n} G_0 - \lambda \delta^k_k$$
$$K_1 = \mu_n G_1 - 2\mu$$
$$K_2 = \mu_n t_n G_2 .$$

We observe that in the Reiner-Rivlin fluid, we may include K_0 in p with no loss of generality. Further, we observe that the form of K_1 is the same for both the Stokesian and Reiner-Rivlin fluids.

The form of the coefficients G_γ . - We consider only isotropic fluids and assume G_γ analytic. Then G_γ can be placed in the form,

$$G_\gamma = \sum_{I,J,K=0}^{\infty} G_{\gamma IJK} (B\,\mathrm{I})^I (B^2\,\mathrm{II})^J (B^3\,\mathrm{III})^K$$

where I , II , III , are the elementary symmetric functions of the rate of deformation tensor.

The $G_{\gamma IJK}$ are dimensionless functions of $\frac{p}{\bar{p}}, \frac{\Theta}{\bar{\Theta}}$ in the case of the Stokesian fluid; and additionally of $\frac{\bar{p} t_n}{\mu_n}$ in the case of the Reiner-Rivlin fluid. $\bar{p}$ is simply a reference pressure; $\bar{\Theta}$ a reference temperature.

We note that

$$\lambda = ABG_{0100} \tag{3-5}$$
$$2\mu = ABG_{1000}$$

in both the Reiner-Rivlin and Stokesian fluids.

IV. GENERAL EFFECTS OF NON-LINEAR THEORY

The extra stress tensor in extensio. - In order to examine clearly the general effects of the non-linear theory, we write the components of the extra stress tensor in extensio, thus, from (3-3),

$$
\begin{aligned}
v^1_1 &= \lambda d^k_k + 2\mu d^1_1 + K_0 + K_1 d^1_1 + K_2 d^1_k d^k_1 \\
v^2_2 &= \lambda d^k_k + 2\mu d^2_2 + K_0 + K_1 d^2_2 + K_2 d^2_k d^k_2 \\
v^3_3 &= \lambda d^k_k + 2\mu d^3_3 + K_0 + K_1 d^3_3 + K_2 d^3_k d^k_3 \qquad (4\text{-}1) \\
v^1_2 &= 2\mu d^1_2 + K_1 d^1_2 + K_2 d^1_k d^k_2 \\
v^1_3 &= 2\mu d^1_3 + K_1 d^1_3 + K_2 d^1_k d^k_3 \\
v^2_3 &= 2\mu d^2_3 + K_1 d^2_3 + K_2 d^2_k d^k_3 .
\end{aligned}
$$

Kelvin Effect. - The terms $K_0 + K_1 d^1_1$, $K_0 + K_1 d^2_2$, and $K_0 + K_1 d^3_3$ of (4-1) represent the Kelvin effect, a phenomenon that occurs in the theory of non-Newtonian fluids that is not explained in the classical theory. If a hydrostatic pressure equal to the negative of K_0 is wanting, the fluid will tend to change its volume. In the case of an incompressible fluid, an arbitrary hydrostatic pressure may be added to the stress and no new effect is noted over the classical case. This effect has been termed the Kelvin effect by Truesdell[1] after Kelvin who predicted it theoretically[2].

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[1] Ibid.

[2] W. Thomson (Kelvin) and P.G. Tait, Treatise on Natural Philosophy (2nd ed. 2nd half; Cambridge: University Press, 1883).

Poynting Effect. - Each of the components $\mathcal{V}_1^1$, $\mathcal{V}_2^2$ and $\mathcal{V}_3^3$ of the extra stress tensor contains a term of the form $K_2 d_h^1 d_1^k$ etc. Hence normal tensions must be supplied in the planes of motion; if such tensions are wanting, the material will tend to change its length. This phenomenon occurs in the theory of non-Newtonian fluids, without regard for compressibility and we call this the Poynting effect, following Truesdell[3]. Poynting first noted this lengthening[4]. He stated that "when a wire is sufficiently loaded to be straightened, it is lengthened by twisting by an amount proportional to the square of the twist and with a given number of turns, inversely as the length." Poynting later reported[5] the results of measurements of diameter contraction, which verified his predictions.

The effect of pressure on the non-linear theory. - Let us write the extra tensor tensor in the form,

$$\mathcal{V}_j^i = \mathcal{F}_0 \delta_j^i + \mathcal{F}_1 d_j^i + \mathcal{F}_2 d_k^i d_j^k$$

where the $\mathcal{F}_r$ are defined by the corresponding coefficients in (3-2); then the general equations of motion of the steady

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[3]Truesdell, "A new definition of a fluid. I.," XXIX.

[4]J. H. Poynting, "On pressure perpindicular to the shear planes in finite pure stress and the lengthening of loaded wires when twisted." Proceedings of the Royal Society, LLXXII, (1909), 546-599.

[5]J.H. Poynting, "On the changes in the dimensions of a steel wire when twisted and the pressure of distortional waves in steel." Proceedings of the Royal Society, LXXXVI, (1912), 534-561.

state flow of an incompressible fluid may be written in the form

$$\rho f_j + \rho \ddot{x}_j = \left[-p\delta^i_j + \eta_1 d^i_j + \eta_2 d^i_k d^k_j \right]_{,i} . \qquad (4\text{-}2)$$

Let η_1 be constant and let u_i and $\tilde{p}$ be the velocity and pressure which satisfy the corresponding Navier-Stokes equation, that is, let

$$\rho f_j + \rho \dot{u}_j = \left[-\tilde{p}\,\delta^i_j + \eta_1 d^i_j \right]_{,i} .$$

We briefly discuss the conditions under which u_i also satisfies (4-2). To this end we divide the pressure in the latter case into two parts, thus $p = \tilde{p} + P$. Then u_i will satisfy (4-2) whenever

$$P_{,j} = \left[\eta_2 (d^i_k d^k_j) \right]_{,i} . \qquad (4\text{-}3)$$

Generally, the right side of (4-3) may or may not be a gradient. When it is, we can obtain solutions of this type.

Three possibilities occur from (4-3): 1) $\eta_2 = 0$, in which case $p = \tilde{p}$ and the flow is Newtonian; 2) there exists a non-trivial and additional pressure gradient, $P_{,j}$, which satisfies (4-3) and must be added to $\tilde{p}$ in order for u_i to be a solution also in the non-linear case; and 3) there does not exist a function, $P_{,i}$, satisfying (4-3). In this case, secondary flows may have to be introduced in order to solve the non-linear flow, if, indeed, solutions exist at all in a particular case.

It is further noted that a special case of 2) above occurs when $(d^i_k d^k_j)_{,i} = 0$.

V. SIMPLE POISEUILLE FLOW

General. - This flow was first introduced by Euler in 1757. It is considered herein to demonstrate the use of inversion of series to obtain solutions.

Let there be two infinite parallel planes, the lower one fixed, the upper moving parallel to itself at a steady speed, and assume that the fluid adheres to the boundary. We take the x -axis in the direction of motion of the upper plane and the y -axis perpindicular to the planes. Steady state is assumed.

Then

$$\dot{x}(i) = \dot{x}^i = (g(y), 0, 0) \qquad \ddot{x}(i) = 0$$

$$d(ij) = \begin{bmatrix} 0 & \frac{1}{2}g' & 0 \\ \frac{1}{2}g' & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with elementary symmetric functions

$$\mathrm{I} = \mathrm{III} = 0 \;, \quad \mathrm{II} = -\frac{1}{4}g'^2$$

and

$$K_r = f(-\tfrac{1}{4}g'^2) = f(y).$$

The physical components of the stress tensor are given by

$$t(ij) = -p\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \mu g' \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ K_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{K_1 g'}{2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{K_2 g'^2}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The Kelvin and Poynting effects are observed even in the case of isochoric flow. A pressure must be supplied perpindicular to the planes; if this pressure be wanting, the upper plane will rise or fall, depending on the sign of $\mathcal{T}_2$.

The equations of motion. - The equations of motion are

$$
\begin{aligned}
0 &= \frac{\partial}{\partial x}\left(-p + K_0 + \tfrac{1}{4} g'^2 K_2\right) + \frac{\partial}{\partial y}\left(\tfrac{1}{2} g'[K_1 + 2\mu]\right) \\
0 &= \frac{\partial}{\partial x}\left(\tfrac{1}{2} g'[K_1 + 2\mu]\right) + \frac{\partial}{\partial y}\left(-p + K_0 + \tfrac{1}{4} g'^2 K_2\right) \qquad (5\text{-}1 \\
0 &= \frac{\partial p}{\partial z}.
\end{aligned}
$$

The solution. - From $(5\text{-}1)_2$, we have

$$\frac{\partial^2 p}{\partial x \partial y} = 0.$$

hence, from $(5\text{-}1)_1$

$$\frac{\partial^2 (g' \mathcal{T}_1)}{\partial y^2} = 0 \qquad (5\text{-}2)$$

where,

$$\mathcal{T}_1 = ABG_1 = K_1 + 2\mu.$$

Integrating (5-2), we obtain

$$t\langle 12\rangle = \tfrac{1}{2} g' \mathcal{T}_1 = A_1 y + A_2 \qquad (5\text{-}3)$$

the resistance per area of the flow.

Again from $(5\text{-}1)_1$,

$$-p = A_1 x + \phi(y)$$

and from $(5\text{-}1)_2$

$$-p + K_0 + \frac{1}{4} g'^2 K_2 = -A_3 + \psi(x).$$

Hence

$$t(11) = t(22) = A_2 = \frac{1}{2} g' \mathcal{F}_1 (-\tfrac{1}{4} g'^2) \qquad (5\text{-}4)$$

wherein A_α are constants of integration.

Rivlin[1] considered the compressible case and, further assumed $A_1 = 0$, obtaining the resistance per area

$$t(12) = A_2 = \frac{1}{2} g' \mathcal{F}_1 (-\tfrac{1}{4} g'^2)$$

or,

$$t(12) = a_1 g' + a_2 g'^3 + a_3 g'^5 + \cdots\cdots\cdots$$

Truesdell[2] has pointed out that, depending on how many terms we take in this expansion, there will be an odd (or infinite) number of functions g (i.e., a certain velocity profile) which will yield the same resistance per area. However, we shall assume that g is an analytic function and furthermore that $t(12)$ is a monotonic function of g'. This will permit inversion of the above series, obtaining g' as a function of y. We then integrate term by term to obtain the function $g(y)$. This will be accomplished shortly.

The rectilinear shearing flow of a compressible

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[1] Rivlin, "The hydrodynamics ......," CXCIII.

[2] Truesdell, "A new definition of a fluid, I.," XXIX

non-Newtonian fluid was first suggested by Reiner[3] and Rivlin[4] and elaborated upon by Truesdell[5]. Further work by Truesdell[6] treats the insufficiency of the shearing forces in the non-linear case. He also considered the rectilinear shearing flow of a compressible Stokesian fluid and reported "the possible velocity profiles $\mathfrak{g}$ now depend upon the pressure $\mathfrak{p}$ as well as the resistance" and that "as the pressure is reduced, the resistance is diminished as should be expected in a gas. Both in the classical theory and in the second approximation, we have $\mu_0 \mathfrak{g}' \mathfrak{f}_{1000} = 2A_2$, but in the third approximation, the resistance is diminished by $\frac{\mu_0^3 \mathfrak{g}'^3 \mathfrak{f}_{1000}}{8\mathfrak{p}^2}$ from the value predicted by the classical theory for the same velocity profile $\mathfrak{g}$."

Truesdell's statements regarding the possible profiles of $\mathfrak{g}$ are certainly correct. However, since we are dealing with physical phenomena, one would expect a more unique behavior of the function $\mathfrak{g}$.

Returning to (5-3), we rewrite,

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[3] Reiner, "A mathematical theory of dilatancy," LXVII.

[4] Rivlin, "The hydrodynamics ......," CXCIII.

[5] Truesdell, "A new definition of a fluid. I.," XXIX.

[6] C. Truesdell, "The mechanical foundations of elasticity and fluid dynamics," The Journal of Rational Mechanics & Analysis, I, (1952), 125-300.

$$\frac{1}{2} g' \tau_1 = -A_1 y + A_2 \tag{5-3}$$

where the left hand side is a power series in g'. From the assumptions regarding analyticity and monotonicity, it is possible to invert (5-3) to obtain g' as a power series in y.

In either the Reiner-Rivlin or the Stokesian fluids, we have

$$\tau_1 = ABG_1 = \mu_n G_1$$

where

$$G_1 = \sum_{J=0}^{\infty} G_{1oJo} \left[-\left(\frac{Bg'}{2} \right)^2 \right]^J.$$

We assume the G_{IoJo} constants.

Then

$$\tau_1 = \mu_n \left(a_0 - \frac{a_1 B^2}{4} g'^2 + \frac{a_2 B^4}{16} g'^4 - \cdots\cdots \right)$$

and from (5-3) and (3-5),

$$y - \frac{A_2}{A_1} = -\frac{\mu}{A_1} g' + \frac{\mu_n a_1 B^2}{8 A_1} g'^3 - \frac{\mu_n a_2 B^4}{32 A_1} + \cdots\cdots \tag{5-5}$$

$$A_1 \neq 0 .$$

Reverting [7] (5-5), we obtain

$$g'(y) = -\frac{1}{\mu}(A_1 y - A_2) - \frac{\mu_n B^2 a_1}{8\mu^4}(A_1 y - A_2)^3 - \frac{\mu_n B^4(3\mu_n a_1^2 - 2\mu a_2)}{64\mu^7}(A_1 y - A_2)^5 - \cdots\cdots$$

Whereupon, integrating, term by term,

$$g(y) = A_4 + \frac{A_2}{\mu}y - \frac{A_1}{2\mu}y^2 - \frac{\mu_n B^2 a_1}{32A_1\mu^4}(A_1 y - A_2)^4 + \cdots\cdots \quad (5-$$

Again we observe the effects of assumptions in the classical case. For example, if A_1 is assumed to be zero, we immediately obtain a linear profile.

In (5-6), the condition of adherence to the bottom plane gives $A_4 = 0$; that of adherence to the top plane will yield a value of A_1 or A_2. A further boundary condition is necessary, say, g'' at the upper plane, in order to determine the profile uniquely.

<u>The pressure</u>. - The pressure was determined in (5-4), namely,

$$p = K_0 + \frac{1}{4}g'^2 \mathcal{T}_2 - A_1 x - A_3 .$$

In the case of an incompressible fluid, K_0 can be disregarded as previously discussed. In the case of a compressible fluid, K_0 determines the Kelvin effect. In both cases, it

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[7] Quite formally, if,

$$y = ax + bx^2 + cx^3 + dx^4 + ex^5 + \cdots\cdots$$

$$x = \frac{1}{a}y - \frac{b}{a^3}y^2 + \frac{2b^2 - ac}{a^5}y^3 + \frac{5abc - a^2d - 5b^3}{a^7}y^4 + \cdots\cdots$$

observed that the pressure is linearly dependent on the downstream variable.

Finally, in the case of isochoric rectilinear shear of a compressible Stokesian fluid, terms of the form $B^{2n} = \left(\frac{\mu_n}{p}\right)^{2n}$ which appear in the explicit expression for $g(y)$ in (5-6), indicate, as Truesdell has often stressed, the appropriateness of the Stokesian theory for low pressure phenomena in gases. In this case, there is no longer a general explicit expression for the pressure since in general, $\mathcal{F}_0, \mathcal{F}_2$ and g are power series in $\frac{1}{p}$. However, in the case that $\mathcal{F}_0$ and $\mathcal{F}_2$ are constant,

$$g = \frac{A_2}{\mu} y - \frac{A_1}{2\mu} y^2$$

and the expression for the pressure remains valid.

VI. POISEUILLE FLOW IN A CIRCULAR PIPE

General. - We briefly consider laminar flow through a cylindrical pipe under steady state conditions in order to derive an extension of the classical efflux law. We take the axis of the pipe as the z -axis in a cylindrical coordinate system r, ϕ, z . Adherence to the boundary is assumed. The velocity field is given by

$$\dot{x}(i) = (0, 0, g(r))$$

and the acceleration is zero.

The equations of motion. - The equations of motion are given by

$$0 = \frac{\partial}{\partial r}(-p + \tau_0) + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{4} r g'^2 \tau_2\right)$$

$$0 = \frac{\partial p}{\partial \phi}$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial r}\left(\frac{1}{2} g' \tau_1\right). \qquad (6\text{-}1)$$

The form of the pressure. - From $(6\text{-}1)_3$, we deduce

$$\frac{\partial p}{\partial z} = h(r)$$

or

$$p = z h(r) + k(r)$$

and therefore

$$\frac{\partial p}{\partial r} = z h'(r) + k'(r)$$

but from $(6\text{-}1)_1$, we note that

$$\frac{\partial p}{\partial r} = f(r)$$

hence $h'=0,\ h=A_1$, and the pressure gradient is therefore constant downstream.

Again from $(6\text{-}1)_3$,

$$\frac{1}{2}g'\mathcal{T}_1\left(-\frac{1}{4}g'^2\right) = -A_1 r. \tag{6-2}$$

Integrating $(6\text{-}1)_1$, by parts, we obtain

$$p-\mathcal{T}_0 = -A_1 z + \frac{1}{4}g'^2\mathcal{T}_2 + \int\frac{1}{4}\frac{g'^2\mathcal{T}_2}{r}\,dr$$

and substituting for g'^2 from (6-2), we obtain

$$p = \mathcal{T}_0 - A_1 z + \frac{1}{4}g'^2\mathcal{T}_2 + A_1^2\int r\frac{\mathcal{T}_2}{\mathcal{T}_1}\,dr. \tag{6-3}$$

<u>The flux.</u> - The main flux through the tube of an incompressible fluid is given by

$$Q = \int_0^a 2\pi r g\,dr$$

Integrating by parts, we obtain

$$Q = 2\pi\rho\left\{\left(\frac{r^2 g}{2}\right)\Big|_0^a - \int_0^a\frac{r^2 g'}{2}\,dr\right\}.$$

The first term in the bracket vanishes because of the condition of adherence, and therefore

$$Q = 2\pi\rho A_1\int_0^a\frac{r^3}{\mathcal{T}_1}\,dr.$$

If $\mathcal{T}_r$ is constant, then from (6-2), we may write

$$g' = -\frac{A_1 r}{\mu}$$

$$g = \frac{A_2(a^2-r^2)}{2\mu}$$

a parabola, the classical result of Poiseuille.

Further, from (6-3)

$$p = -A_1 z + \frac{3A^2 r^2 F_2}{4\mu^2} + A_3 .$$

Defining the inlet pressure,

$$p_i \equiv p\Big|_{\substack{r=0\\z=0}} = A_3$$

and the outlet pressure

$$p_o \equiv p\Big|_{\substack{r=0\\z=\ell}} = -A_1 \ell + p_i$$

where ℓ is the length of the pipe,

we have

$$A_1 = \frac{p_i - p_o}{\ell}$$

and, the flux

$$Q = \frac{\pi \rho a^4}{8\mu}\left(\frac{p_i - p_o}{\ell}\right). \qquad (6\text{-}4)$$

This expression for the flux was first given experimentally by Poiseuille [1,2,3] and is termed the Hagen-Poiseuille Law of

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[1] J.L.M. Poiseuille, "Experimental research on the motion in tubes of very small diameter," Académie des Sciences Comptes Rendus, XI, (1840), 961-967, 1041-1048.

[2] Ibid, XII, (1841), 112-115.

[3] J.L.M. Poiseuille, "Experimental research on the motion in tubes of very small diameter," Mémoire des Savants Étrangers, 19: (1846), 433-544.

Efflux from a pipe. The formal derivation is due to Rivlin[4]. A generalization to this law is derived below for the non-linear case.

We observe that the velocity distribution and the mass flux are unchanged from the classical case in the second order ($\mathcal{F}_{z}$ = constant) theory.

The solution by inversion. - We return to the differential equation, (6-2),

$$\frac{1}{2} g' \mathcal{F}_1 \left(-\frac{1}{4} g'^2\right) = -A_1 r. \tag{6-2}$$

By methods similar to those in the previous chapter, we derive

$$g(r) = \frac{A_1}{2\mu}(a^2 - r^2) + \frac{\mu_n A_1^3 B^2 a_1}{32\mu^4}(a^4 - r^4) + \cdots\cdots$$

The mass flux in the case of a non-Newtonian incompressible fluid is given by

$$Q = 2\pi\rho \int_0^a r g \, dr$$

$$= \pi\rho \left[\frac{A_1 a^4}{4\mu} + \frac{\mu_n B^2 A_1^3 a_1 a^6}{48\mu^4} + \cdots\cdots \right].$$

Computing only the first two terms, and taking $A = \frac{p_i - p_o}{l}$, as before, we obtain

$$Q = \frac{\pi\rho a^4}{8\mu}\left(\frac{p_i - p_o}{l}\right) + \frac{\pi\rho\mu_n B^2 a_1 a^6}{48\mu^4}\left(\frac{p_i - p_o}{l}\right)^3$$

an extension of the Hagen-Poiseuille Efflux Law, which is

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[4] Rivlin, "The Hydrodynamics....," CXCIII

more conveniently written as

$$Q = \frac{\pi \rho a^3}{8\mu}\left[\left(\frac{a(p_i - p_o)}{\ell}\right) + \frac{\mu_o B^2 a_1}{6\mu^3}\left(\frac{a(p_i - p_o)}{\ell}\right)^3\right].$$

Poiseuille used a "pipe" of radius ranging from 0.007 to 0.325 mm. The new law predicts that if similar experiments were performed with $\frac{a(p_i - p_o)}{\ell}$ sufficiently large, yet the flow still laminar, that a measure of the efflux would be substantially different from that given in the classical efflux law.

In general, $g(r)$ will be an even power series in $\frac{p_i - p_o}{\ell}$, and Q will be an odd power series. The classical Hagen-Poiseuille Efflux Law is valid whenever

$$a\frac{(p_i - p_o)}{\ell} \ll 1.$$

VII. STRUCTURE OF SHOCK WAVES

General. - The existence and structure of shock waves in the non-linear continuum theory was established for the fluids described herein jointly by Gilbarg and the author[1]. In addition therein, we computed shock thicknesses in monotonic gases and compared these results with those of Thomas[2] (continuum theory) and Mott-Smith[3] and Zoller[4] (kinetic theory); and further demonstrated that the basic methods of extending the theory to polyatomic gases remains unchanged in the continuum theory provided that fluid moduli can be determined.

That portion of this joint work pertaining to the structure of shock waves in the non-linear theory is included as it relates to the topic of this dissertation.

The steady one-dimensional flow of a trivariate, heat conducting fluid with non-linear viscosity is considered in the absence of external forces. The flow approaches finite limit values at $x = \pm\infty$. Such flows display the character of a shock

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[1]D. Gilbarg & D. Paolucci, "The structure of shock waves in the continuum theory of fluids," Journal of Rational Mechanics and Analysis, II, (1953), 617-642).

[2]L.H. Thomas, "Note on Becker's theory of the shockfront," Journal of Chemical Physics, XII, (1944), 449-452.

[3]H.M. Mott-Smith, "The solution of the Boltzmann equation for a shock wave," Physical Review, LXXXII, (1951), 256-274.

[4]K. Zoller, "On the structure of shock waves", Zeitshrift für Physik, CXXX, (1951), 1-38.

wave in that they differ from their end states at $X = \pm\infty$ only in a small interval of rapid transition.

The velocity field is given by $\dot{x} = u$ and the rate of deformation as $\frac{\partial u}{\partial x} = u'$, where the prime will denote differentiation with respect to X throughout this chapter.

Equations of motion. - The equation of continuity, Cauchy's first law, the energy equation, Fourier's Law and the extra stress, respectively, take the form

$$
\begin{aligned}
&(\rho u)' = 0 \\
&\rho u u' = t' \\
&\rho u \varepsilon' = t u' - q' \\
&q = -\chi \theta' \\
&v = t + p
\end{aligned}
\tag{7-1}
$$

Elimination of t and q from (7-1) and integration of the resultant system yields

$$
\begin{aligned}
&\rho u = m \\
&\rho u^2 + p - v = P \\
&\rho u\left[\frac{u^2}{2} + \varepsilon + \frac{p}{\rho}\right] - v u - \chi\theta' = E
\end{aligned}
\tag{7-2}
$$

where m, P and E are constants of integration (7-2) expresses, respectively, the conservation of mass, momentum and energy. Eliminating ρ and redefining

$$a = \frac{P}{m}, \quad b = m, \quad c = \frac{E}{m} - \frac{P^2}{2m^2}$$

we obtain

$$\begin{aligned} \tau &= p + b(u-a) \\ \chi\,\theta' &= b\left[\varepsilon - \tfrac{1}{2}(u-a)^2 - c\right] \end{aligned} \tag{7-3}$$

The form of the viscous stress. - We propose to study the effect of non-linear viscosity on the theory of the shock profile, an application naturally suggested by the large velocity gradient present in the shock front.

Of the possible generalizations to fluids obeying a non-linear stress law, we select one in which the viscous stress may be a non-linear function of the velocity gradient, but remains independent of temperature gradients. To fix our ideas, we choose the specific volume, v, and the temperature, θ, as variables of state and define the viscous stress $\tau = \tau(u', v, \theta)$ with the following properties:

a. $\tau(0, v, \theta) = 0$ for all v, θ

b. $\frac{\partial \tau}{\partial u'}(0, v, \theta) = \mu(v, \theta) > 0$ for all v, θ

Property a. asserts the essential characteristic of fluids that the viscous stress vanishes when there is no velocity gradient. Property b. states that for small velocity gradients, the stress is approximately given by the linear law and the fluid essentially becomes the classical one.

For convenience, we consider the function $\tau = \tau(u', u, \theta)$ which we identify with $\tau(u', v, \theta)$ by means of $(7\text{-}2)_1$ and re-write (7-3)

$$v(u', u, \theta) = p + b(u-a)$$
$$\varkappa \theta' = b\left[\mathcal{E} - \tfrac{1}{2}(u-a)^2 - c\right]. \qquad (7\text{-}4)$$

For convenience in exposition, but without essential loss of generality, we limit our discussion to perfect gases. However, the same proof carries over as well to fluids obeying the general equations of state considered by Gilbarg[5] in proofing existence in the linear case. For these gases,

$$p = R\Theta\rho, \quad \mathcal{E} = c_v\theta, \quad R = c_p - c_v$$

where c_p, c_v, are the specific heats per unit mass at constant pressure and volume, respectively. We set $\gamma = \frac{c_p}{c_v}$, as usual, then define $\delta = \frac{1}{2}(\gamma - 1)$.

Following Becker[6], we introduce dimensionless variables by means of the substitutions

$$\omega = \frac{m}{P}u, \quad \beta = \frac{p}{P}, \quad \bar{\Theta} = \frac{m^2 R\Theta}{P^2} = \beta\omega$$

and also define

$$\alpha = \frac{2Em^2}{P^2} - 1, \quad \bar{\varkappa} = \frac{\varkappa}{c_v m}$$

and the function

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[5] D. Gilbarg, "The existence and limit behavior of the one-dimensional shock layer," American Journal of Mathematics, LXXII, (1951), 256-274.

[6] R. Becker "Stossvelle und Detonation," Zeitschrift für Physik, VIII, (1921-22), 331-362.

$$\varphi = \nu\left(u'(\omega',\omega,\bar{\theta}),\ u(\omega',\omega,\bar{\theta}),\ \theta(\omega',\omega,\bar{\theta})\right)$$
$$\equiv \varphi(\omega',\omega,\bar{\theta}).$$

After the above substitutions, (7-4) becomes

$$\varphi(\omega',\omega,\bar{\theta}) = \omega + \frac{\bar{\theta}}{\omega} - 1 \equiv M(\omega,\bar{\theta})$$
$$\bar{\varkappa}\,\bar{\theta}' = \bar{\theta} - \delta\left[(1-\omega)^2 + \alpha\right] \equiv L(\omega,\bar{\theta}) \qquad (7\text{-}5)$$

φ carries over the properties of $\nu(u',u,\theta)$, namely

a'. $\varphi(0,\omega,\bar{\theta}) = 0$ for all $\omega, \bar{\theta}$

b'. $\dfrac{\partial\varphi}{\partial\omega'}(0,\omega,\bar{\theta}) \equiv \bar{\mu}(\omega,\bar{\theta}) > 0$ for all $\omega, \bar{\theta}$.

By the implicit function theorem, these properties have as an immediate consequence the existence of a function $\psi(\omega,\bar{\theta})$ such that

$$\omega' = \omega'\left(M(\omega,\bar{\theta}),\omega,\bar{\theta}\right) \equiv \psi(\omega,\bar{\theta})$$

in some neighborhood of the parabola $M=0$, and such that $\psi=0$ on this curve. Naturally the extent of the domain of definition of $\psi(\omega,\bar{\theta})$ cannot be deduced from the purely local properties such as a' and b'.

In the ω-$\bar{\theta}$ plane (Z-plane), let the two parabolas, $M=0$ and $L=0$ intersect in the points $Z_0=(\omega_0,\bar{\theta}_0)$, $Z_1=(\omega_1,\bar{\theta}_1)$ with $\omega_0 > \omega_1$. We see that such points could correspond to initial and final states of a shock wave, since Z_0 and Z_1 would simultaneously satisfy (7-2) and therefore satisfy the Rankine-Hugoniot shock conditions, namely,

$$\rho_0 w_0 = \rho_1 u_1 = m$$

$$p_0 + \rho_0 u_0^2 = p_1 + \rho_1 u_1^2 = P$$

$$\varepsilon_0 + \frac{1}{2} u_0^2 + \frac{p_0}{\rho_0} = \varepsilon_1 + \frac{1}{2} u_1^2 + \frac{p_1}{\rho_1} = E$$

where the values of u_0, u_1, etc are obtained in the obvious manner.

Equivalent conditions for existence. - We will now reduce (7-5) to a form wherein the techniques developed by Gilbarg[7] may be applied.

We make the further assumption that $\psi(\omega, \bar{\theta})$ is defined throughout the region, R, (see figure 1.) bounded by the parabolas.

Since $\psi = 0$ implies $\omega' = 0$, which in turn implies $M = 0$ (property a'), we observe that $\psi = 0$ only on the parabola $M(\omega, \bar{\theta}) = 0$. Therefore $\psi(\omega, \bar{\theta})$ is of one sign throughout R. Furthermore this sign is negative since

$$\psi_{\bar{\theta}} \Big|_{M=0} = - \frac{\varphi_{\bar{\theta}} - M_{\bar{\theta}}}{\varphi_{\omega'}} \Big|_{\substack{\omega'=0 \\ M=0}} = \frac{M_{\bar{\theta}}}{\varphi_{\omega'}} \Big|_{\substack{\omega'=0 \\ M=0}} > 0$$

If we can prove therefore the existence of an integral curve of the system

$$\begin{aligned} \omega' &= \psi(\omega, \bar{\theta}) \\ \theta' &= \frac{L(\omega, \bar{\theta})}{\bar{\chi}} \end{aligned} \qquad (7\text{-}6)$$

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[7] Gilbarg, "The existence ......," LXXII.

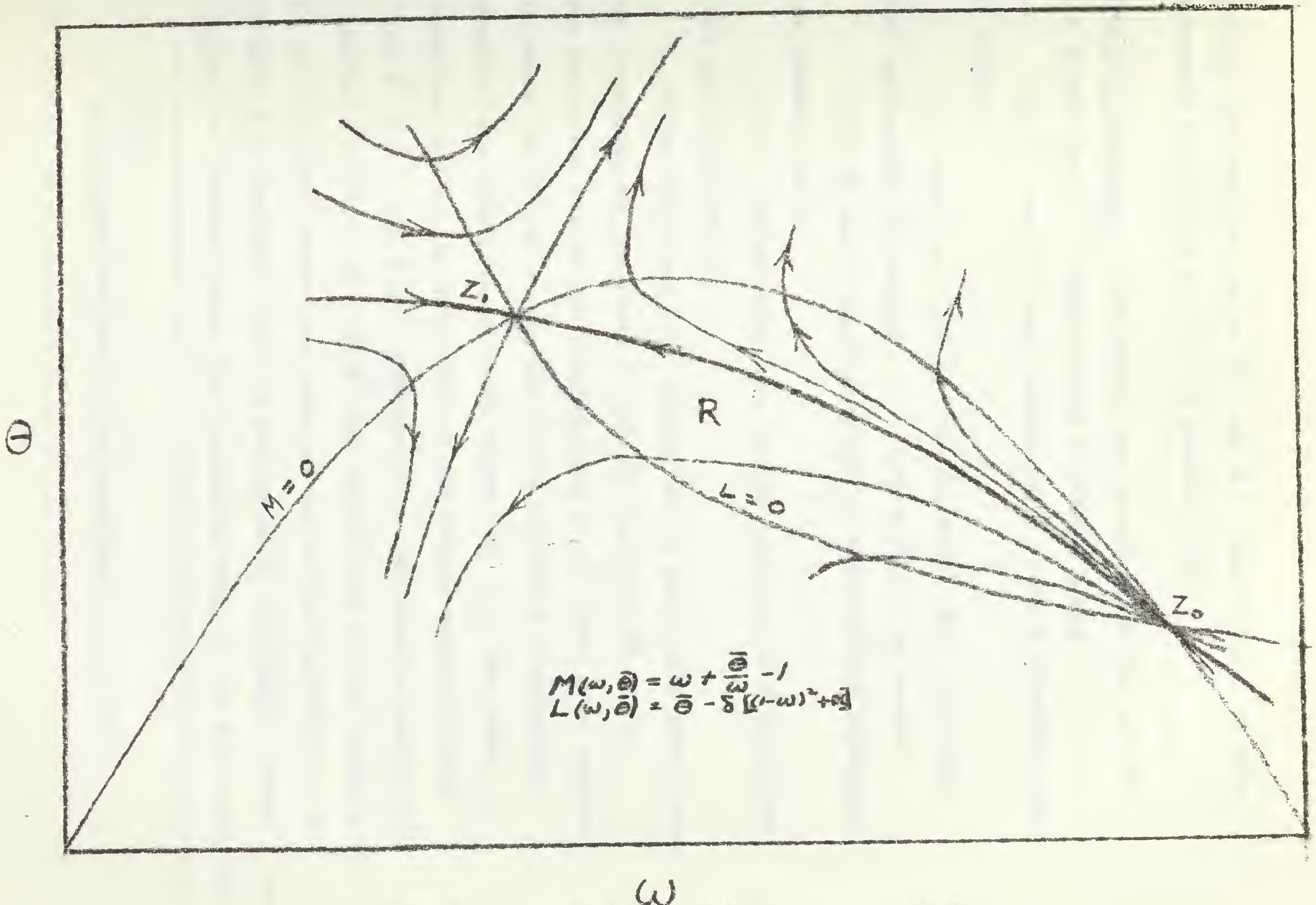

Figure 1. Direction field in plane

joining the singular point Z_0 and Z_1, then clearly we have succeeded in proving it for the system (7-5).

<u>Proof of existence.</u> - The proof, patterned after the one for the linear case [8,9] rests on the following facts:

(i) Z_0 and Z_1 are, respectively, node and saddle point of (7-6);

(ii) there is a unique integral curve $S(x)$ of (7-6) that approaches the saddle, Z_1, from within R as $x \to \infty$.

(iii) S has a negative slope in R and cannot intersect the boundary of R.

(iv) S cannot terminate in the interior of R and therefore must approach Z_0.

Steps (iii) and (iv) follow, as in the linear case, from our knowledge of the signs of $M(\omega,\bar{\theta})$ and $L(\omega,\bar{\theta})$ in R. The parabolas divide the neighborhood of Z_1 into four regions in which the slopes of the direction field are alternately positive and negative, and (see figure 1.) in R the slope is negative, and in the direction of the second quadrant since $M(\omega,\bar{\theta}) < 0$, $L(\omega,\bar{\theta}) > 0$ throughout R. We observe now by considering the signs of $L(\omega,\bar{\theta})$ and $M(\omega,\bar{\theta})$ that on the boundary of R, all integral curves are directed out of the region as x increases and therefore into the region as x decreases. Hence if we follow the integral curve $S(x)$ backwards from Z_1 in the direction

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[8] <u>Ibid.</u>

[9] Gilbarg and Paolucci, "The structure ....," II

of decreasing x (and increasing ω), we see that it can never leave R and must approach the other singular point as $x \to -\infty$.

It therefore remains only to discuss the nature of the singular points at Z_0 and Z_1 and thereby to establish (i) and (ii). The characteristic equation of (7-6) at Z_0 and Z_1 is

$$\left| \begin{matrix} \psi_\omega - \epsilon & \psi_{\bar{\theta}} \\ \left(\frac{L}{\bar{\chi}}\right)_\omega & \left(\frac{L}{\bar{\chi}}\right)_{\bar{\theta}} - \epsilon \end{matrix} \right|_{z=z_0, z_1} = 0$$

We observe that

$$\psi_\omega \Big|_{Z_i} = - \frac{\phi_\omega - M_\omega}{\phi_{\omega'}} \Big|_{\substack{\omega'=0 \\ Z_i}} = \frac{M_\omega(Z_i)}{\bar{\mu}(Z_i)} \qquad i = 1, 2$$

similarily

$$\psi_\theta \Big|_{Z_i} = \frac{M_{\bar{\theta}}(Z_i)}{\bar{\mu}(Z_i)}$$

while

$$\left(\frac{L}{\bar{\chi}}\right)_\omega \Big|_{Z_i} = \frac{L_\omega(Z_i)}{\bar{\chi}(Z_i)}$$

and

$$\left(\frac{L}{\bar{\chi}}\right)_{\bar{\theta}} \Big|_{Z_i} = \frac{L_{\bar{\theta}}(Z_i)}{\bar{\chi}(Z_i)}$$

Consequently the characteristic equation is precisely that given in the linear case, that is,

$$0 = \epsilon^2 - \left(\frac{M_\omega}{\bar{\mu}} + \frac{L_{\bar{\theta}}}{\bar{\chi}}\right)\epsilon + \left(\frac{M_{\bar{\theta}} L_{\bar{\theta}}}{\bar{\mu}\,\bar{\chi}}\right)\left(\frac{M_\omega}{M_{\bar{\theta}}} - \frac{L_\omega}{L_{\bar{\theta}}}\right)$$

By inserting the values of M and L and subsequently evaluating at Z_0 and Z_1 , and after simple but tedious

algebraic computation, it is not difficult to see that the roots of the characteristic equation are real and of opposite sign at Z_1 and real and of the same sign at Z_0. Thus Z_1 is a saddle point and Z_0 and unstable node. There are exactly two integral curves that approach the saddle Z_1 as $x \to \infty$ and exactly two that approach as $x \to -\infty$. These pairs correspond to the negative and positive roots, respectively of the characteristic equation. Two members of each pair have the same slope, but approach from opposite direction. Thus (i) and (ii) have been demonstrated and the existence established.

Numerical computation. - The numerical calculation of shock thicknesses in the linear fluid under discussion requires somewhat exact knowledge of the viscous stress, $\tau(u', u, \theta)$. In an effort to arrive at an estimate of the influence of non-linear viscosity on the shock profile, we consider a gas for which we assume a quadratic viscous stress of the form

$$\tau = \frac{4}{3}\mu u' + \alpha(v, \theta) u'^2 \qquad (7\text{-}7)$$

where μ is the usual shear viscosity and $\alpha(v,\theta)$ is a coefficient still to be determined. For lack of specific empirical information concerning α, we arbitrarily take a value suggested by the Burnnett approximation of the kinetic theory. In this approximation, the expression for viscous stress in one-dimensional form in the absence of external forces and which contribute to (7-7) is

$$\tau = \frac{4}{3}\mu u' - \frac{\mu^2}{p}\left(\frac{2}{3}\varpi_1 - \frac{14}{9}\varpi_2 + \frac{8}{27}\varpi_6\right)u'^2$$

where ϖ_1, ϖ_2, ϖ_6 are dimensionless quantities whose values depend on the molecular model.

Simply to serve as an example, we take the values for ϖ_1, ϖ_2 and ϖ_3 given by Chapman and Cowling[11]

$$\varpi_1 = \frac{4}{3}\left(\frac{7}{2} - \frac{\theta}{\mu}\frac{d\mu}{d\theta}\right), \quad \varpi_2 = 2, \quad \varpi_6 = 8.$$

Substituting our arbitrary expressions for $\mathbf{v}$ into system (7-5) we obtain

$$\bar{\mu}\,\bar{\omega}' = \frac{\sqrt{1 + 4A\omega\frac{M}{\bar{\Theta}}} - 1}{\frac{2A\omega}{\bar{\Theta}}}$$

$$\bar{\chi}\,\Theta' = \bar{\Theta} - \delta\left[(1-\omega)^2 + \alpha\right]$$

The positive square root is taken for ω' in order that $\omega' = 0$ imply $M = 0$. For purpose of calculating the shock profile, this system can now be treated as was that resulting from the Navier-Stokes equation. If $\mu \sim \Theta^s$, then the expression for A is simply $A = \frac{1}{2}s - \frac{4}{3}$.

In order to compare with previous results obtained from the linear theory, we specialize to the case of Helium for which $s = .647$ and $\frac{\bar{\mu}}{\bar{\chi}} = \frac{8}{15}$. If we designate by $\Delta_{lin.}$ and $\Delta_{quad.}$ the shock thickness according to the linear and non-linear theory, respectively, our sample calculation show:

at $\quad M_0 = 2 \qquad \dfrac{\Delta_{quad.}}{\Delta_{lin.}} = 1.16$

and at $\quad M_0 = 4 \qquad \dfrac{\Delta_{quad.}}{\Delta_{lin.}} = 1.30$.

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[11] S. Chapman & T. Cowling, The Mathematical Theory of Non-uniform Gases. (Cambridge: University Press, 1939)

Thus the shock thickness increases in going from the linear to the non-linear, the percentage increase being greater at the higher Mach numbers. Of course, the significance of these numerical results is relative, since the choice of the form of the viscous stress was arbitrary. In a specific real fluid the effect of non-linearity might be greater or smaller depending on the empirical value of the coefficient of the quadratic term. However, the discussion shows that the potentialities of the continuum theory are by no means limited to the Navier-Stokes Equations.

VIII. STEADY FLOW BETWEEN NON-PARALLEL WALLS

Introduction. - We consider the laminar flow of an incompressible fluid between non-parallel walls, under steady state conditions. We take the intersection of the walls as the z-axis in a cylindrical polar coordinate system and assume adherence to the walls. The coordinate, r , the distance from the z-axis is assumed greater than zero. The equations of the walls are therefore given by $\phi = \pm\phi_1$.

General. - Radial flow and absence of secondary flow is assumed.

$$\dot{x}(i) = (u, 0, 0).$$

Then the equation of continuity takes the form

$$\frac{\partial u}{\partial r} + \frac{u}{r} = 0$$

which is satisfied by

$$r\,u = g(\phi, z)$$

and, finally, if we assume that $\frac{\partial u}{\partial z} = 0$, then

$$u = \frac{g(\phi)}{r}.$$

The inertial terms are therefore given by

$$\ddot{x}(i) = \left(-\frac{g^2}{r^3}, 0, 0\right)$$

and the rate of deformation tensor as

$$d(ij) = \frac{1}{2r^2}\begin{bmatrix} -2g & g' & 0 \\ g' & 2g & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with elementary symmetric functions

$$\mathrm{I} = \mathrm{III} = 0 \qquad \mathrm{II} = -\frac{1}{4r^4}\left(4g^2 + g'^2\right).$$

The stress tensor is written

$$t(ij) = -p\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{\mu}{r^2}\begin{bmatrix} -2g & g' & 0 \\ g' & 2g & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \frac{K_1}{2r^2}\begin{bmatrix} -2g & g' & 0 \\ g' & 2g & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{K_2}{4r^4}\begin{bmatrix} 4g^2+g'^2 & 0 & 0 \\ 0 & 4g^2+g'^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The Poynting effect is observed. As in the case of flow in a pipe, $t(22)$ differs from the pressure by a quantity which is always greater on the edges. Hence, if this pressure is wanting and the walls are terminated at $r = r_1 > 0$, then the emerging fluid will swell.

Equations of motion.

Let $\Theta = \mathrm{II}$.

Then

$$F_r = F_r(\Theta)$$

and the stress components may be written

$$t(11) = -p - \frac{F_1 g}{r^2} + F_2\Theta$$

$$t(22) = -p + \frac{F_1 g}{r^2} + F_2\Theta$$

$$t(33) = -p$$

$$t(12) = \frac{F_1 g'}{2r^2}$$

$$t(13) = t(23) = 0.$$

The equations of motion are

$$-\frac{e g^2}{r^3} = -\frac{\partial p}{\partial r} - \frac{1}{r^2}\frac{\partial(g\tau_1)}{\partial r} + \frac{\partial(\tau_2\theta)}{\partial r} + \frac{1}{2r^3}\frac{\partial(g'\tau_1)}{\partial \phi}$$

$$0 = -\frac{\partial p}{\partial \phi} + \frac{1}{r^2}\frac{\partial(g\tau_1)}{\partial \phi} + \frac{\partial(\tau_2\theta)}{\partial \phi} + \frac{1}{2r}\frac{\partial(g'\tau_1)}{\partial r} \qquad (8\text{-}1)$$

$$0 = -\frac{\partial p}{\partial z}.$$

Differentiating $(8\text{-}1)_1$ with respect to ϕ and $(8\text{-}1)_2$ with respect to r, and subtracting, we obtain

$$-4egg' = -4r\frac{\partial^2(g\tau_1)}{\partial r\partial\phi} + 4\frac{\partial(g\tau_1)}{\partial\phi} + \frac{\partial^2(g'\tau_1)}{\partial\phi^2} \qquad (8\text{-}2)$$

$$-r^2\frac{\partial^2(g'\tau_1)}{\partial r^2} + r\frac{\partial(g'\tau_1)}{\partial r}.$$

In order to separate explicitly the dependence of the flow variables on the space variables, we introduce several substitutions. We first take

$$\theta = \frac{H(\phi)}{4r^4}, \quad H(\phi) \equiv 4g^2 + g'^2. \qquad (8\text{-}3)$$

Then

$$\frac{\partial\theta}{\partial r} = -\frac{4\theta}{r} \qquad \frac{\partial^2\theta}{\partial r^2} = \frac{20\theta}{r^2}$$

$$\frac{\partial\theta}{\partial\phi} = \theta(\ln H)' \qquad \frac{\partial^2\theta}{\partial\phi^2} = \theta(\ln H')'(\ln H)' \qquad (8\text{-}4)$$

$$\frac{\partial^2\theta}{\partial r\partial\phi} = -\frac{4}{r}\theta(\ln H)'.$$

Substituting (8-3) and (8-4) in (8-2) and after some manipulation and gathering of terms, we obtain

$$-4egg' = \mathcal{F}_1''\Theta^2\left[g'(\ln H)'^2 - 16g' + 16g(\ln H)'\right]$$
$$+ \mathcal{F}_1'\Theta\left[2g''(\ln H)' + g'(\ln H')'(\ln H)' - 20g(\ln H)' - 8g'\right]$$
$$+ \mathcal{F}_1\left[4g' + g'''\right] \qquad (8\text{-}5)$$

independent of $\mathcal{F}_2$.

Substituting for Θ, and rewriting, we have

$$\frac{1}{16\pi^8}\frac{\partial^2\mathcal{F}_1}{\partial\Theta^2}P_2(\phi) + \frac{1}{4\pi^4}\frac{\partial\mathcal{F}_1}{\partial\Theta}P_1(\phi) + \mathcal{F}_1 P_0(\phi) + 4egg' = 0 \qquad (8\text{-}6)$$

where

$$P_2(\phi) \equiv \left[g'H'^2 + 16gHH' - 16g'H^2\right]$$

$$P_1(\phi) \equiv \left[-2g''H - g'H'' - 20gH' + 8g'H\right]$$

$$P_0(\phi) \equiv \left[g''' + 4g'\right].$$

<u>Solution</u>. - Suppose now that we take $\mathcal{F}_1$ of the form

$$\mathcal{F}_1 = \sum_{\nu=0}^{\infty} a_\nu \Theta^\nu$$

where the a_ν are constant. Then

$$\frac{\partial\mathcal{F}_1}{\partial\Theta} = \sum_{\nu=1}^{\infty} \nu a_\nu \Theta^{\nu-1}$$

$$\frac{\partial^2\mathcal{F}_1}{\partial\Theta^2} = \sum_{\nu=2}^{\infty} \nu(\nu-1)a_\nu \Theta^{\nu-2}$$

Recalling that $\Theta = -\frac{H(\phi)}{4\pi^4}$ and substituting for $\mathcal{F}_1$ and its derivatives in (8-6), we obtain

$$0 = P_2\sum_{\nu=2}^{\infty}\nu(\nu-1)a_\nu\frac{(-H)^{\nu-2}}{(4\pi^4)^\nu} + P_1\sum_{\nu=1}^{\infty}\nu a_\nu\frac{(-H)^{\nu-1}}{(4\pi^4)^\nu}$$
$$+ P_0\sum_{\nu=0}^{\infty}a_\nu\frac{(-H)^\nu}{(4\pi^4)^\nu} + 4egg'$$

or equivalently,

$$0 = \sum_{\nu=2}^{\infty}\left[\nu(\nu-1)P_2 - \nu P_1 H + P_0 H^2\right] a_\nu \frac{(-H)^{\nu-2}}{(4\pi^4)^\nu} \tag{8-7}$$

$$+ (P_1 - P_0 H)\frac{a_1}{4\pi^4} + P_0 a_0 + 4egg'.$$

Equating the coefficients of the powers of $\left[\frac{1}{4\pi^4}\right]$ on each side of (8-7) we obtain the following equations,

$$P_0 a_0 + 4egg' = 0 \tag{8-8}$$

$$\left[\nu(\nu-1)P_2 - \nu P_1 H + P_0 H^2\right] a_\nu = 0$$

$$\text{for } \nu = 1, 2, 3, \cdots\cdots \tag{8-9}$$

Supposing, in (8-9), that $a_\nu \neq 0$ for all $\nu \geq 1$ then

$$\nu(\nu-1)P_2 - \nu P_1 H + P_0 H^2 = 0 \text{ for all } \nu \geq 1 \tag{8-10}$$

is a system of linear algebraic equations in the **three** **unknowns** P_2, $P_1 H$, $P_0 H^2$.

For any α, β, γ, all unequal, the determinant of coefficients of the system (8-10) is given by

$$\begin{vmatrix} \alpha(\alpha-1) & -\alpha & 1 \\ \beta(\beta-1) & -\beta & 1 \\ \gamma(\gamma-1) & -\gamma & 1 \end{vmatrix} = \begin{vmatrix} \alpha^2 & -\alpha & 1 \\ \beta^2 & -\beta & 1 \\ \gamma^2 & -\gamma & 1 \end{vmatrix}$$

the Vandermonde determinant, which has the value

$$(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$$

and vanishes if and only if at least two of α, β, γ are are equal, contrary to hypothesis. Hence it cannot be the case that all a_ν are non-zero.

Repetition of the above argument demonstrates that all $a_\nu \geq 1$ are zero except for at most two different, say α and β. In this case (8-10) reduces to

$$\begin{aligned} \alpha(\alpha-1)P_2 - \alpha P_1 H &= -P_0 H^2 \\ \beta(\alpha-1)P_2 - \beta P_1 H &= -P_0 H^2 \end{aligned} \tag{8-11}$$

or

$$P_2 = \frac{P_0 H^2}{\alpha\beta}$$

$$P_1 = \frac{P_0 H(\beta+\alpha-1)}{\alpha\beta} = P_2 H(\beta+\alpha-1)$$

where α and β are to be determined.

Now from (8-8), since $a_0 = 2\mu$

$$g''' + 4g' + \frac{2e}{\mu} g g' = 0 \tag{8-12}$$

which is the differential equation of the classical case stuidied by Jeffrey[1], Hamel[2], Harrison[3] and Oseen[4], and which after integration, yields

$$g'^2 = \frac{2e}{\mu}\left[A_2 - \frac{3\mu A_1 g}{e} - \frac{6\mu g^2}{e} - g^3\right] \tag{8-13}$$

where A_1 and A_2 are constants of integration. This equation is

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[1] G.B. Jeffrey, "The two-dimensional steady motion of a viscous fluid," Philosophical Magazine, XXIX, (1915), 455-465.

[2] G. Hamel, "Special motion of viscous fluid," Deutsche Mathematiker Vereinigung, XXV, (1917), 35-60.

[3] W.J. Harrison, "The pressure in a viscous liquid moving through a channel with diverging boundaries," Proceedings of of the Cambridge Society, XIX, (1920), 307-312.

[4] C.W. Oseen, "Exact solutions of hydrodynamic differential equations," Arkiv för Matematik, Stockholm, No. 14, (1927); No. 22, (1928).

further studied by means of elliptical integrals.

We write, in lieu of (8-13),

$$g' = -\left[\frac{2\ell}{\mu}(c_1-g)(c_2-g)(c_3-g)\right]^{1/2}$$

where two of the c_1, c_2, c_3, are constants of integration and

$$c_1 + c_2 + c_3 = -\frac{6\mu}{\ell}$$

This requires at least one of the c_i should have a negative real part. If we write

$$\mathcal{R}_e(c_1) \geqslant \mathcal{R}_e(c_2) \geqslant \mathcal{R}_e(c_3)$$

then

$$\mathcal{R}_e(c_3) \leqslant -\frac{2\mu}{\ell}$$

the sign of equality being valid only when the c_i all have the same real part. The different possible cases have been discussed by Hamel.

Returning to (8-11), if we substitute the value of from (8-8), we have two equations in g, its derivatives and their powers. (8-11) is then a set of diaphantine equations in α and β. I have not been able to determine whether non-zero solutions exist in this case. If such solutions do not exist, it would appear that some different type of motion, possibly including secondary flows, must occur.

<u>The pressure</u>. - Suppose that $\bar{r}_1$ is constant, i.e., equal to $\lambda\mu$. Then by substitution in equations $(8\text{-}1)_{1,2}$, adding, and

collecting terms, we obtain

$$\frac{\partial}{\partial r}(p - \tau_2 \theta) + \frac{1}{r}\frac{\partial}{\partial \phi}(p - \tau_2 \theta) = \frac{Q_1(\phi)}{r^3}$$

where

$$Q_1(\phi) \equiv \mu g'' + 2\mu g' + \rho g^2 \tag{8-14}$$

We may therefore assume that

$$r^3 \frac{\partial}{\partial r}(p - \tau_2 \theta) = 2 Q_2(\phi) \tag{8-15}$$

and

$$r^2 \frac{\partial}{\partial \phi}(p - \tau_2 \theta) = Q_3(\phi) \tag{8-15}$$

with

$$2Q_2 + Q_3 = Q_1. \tag{8-16}$$

Integrating $(8\text{-}15)_{1,2}$ with respect to r and ϕ, we obtain

$$\left.\begin{aligned} p - \tau_2 \theta &= -\frac{Q_2}{r^2} + k_1(\phi) \\ p - \tau_2 \theta &= \frac{1}{r^2}\int Q_3 \, d\phi + k_2(r) \end{aligned}\right\} \tag{8-17}$$

since $\frac{\partial p}{\partial z} = 0$ from $(8\text{-}1)_3$.

Integrating (8-12), we obtain the second derivative of the function g,

$$\mu g'' + 4\mu g + \rho g^2 + \mu A_1 = 0$$

which, when combined with (8-14) and (8-16) yields

$$2Q_2(\phi) + Q_3(\phi) - 2\mu g' + 4\mu g + \mu A_1 = 0.$$

Supposing now that we take, for convenience,

$$Q_2 \equiv -2\mu g - \frac{\mu A_1}{2}$$

and $\quad Q_3 \equiv 2\mu g'$

then (8-17) become

$$p - \mathcal{T}_2\Theta = \frac{2\mu g}{r^2} + \frac{\mu A_1}{2r^2} + k_1(\phi)$$

$$p - \mathcal{T}_2\Theta = \frac{2\mu g}{r^2} + k_2(r)$$

and finally, by comparison

$$p = \frac{2\mu g(\phi)}{r^2} + \frac{\mu A_1}{2r^2} + \mathcal{T}_2\,\Theta + \text{const.}$$

The above analysis demonstrates that the flow of a non-Newtonian fluid in a wedge yields a velocity profile identical to the flow of a Newtonian fluid provided no secondary flow exists. The pressure, however differs from the classical case by an amount $\mathcal{T}_2\,\Theta$. It is noted that no restriction has been placed on $\mathcal{T}_2$, while $\mathcal{T}_1$ was assumed constant in arriving at this conclusion.

IX. THE PROBLEM OF BOUSSINESQ

General. - In 1880, Boussinesq[1] posed and discussed the problem of "how the retarding influence of viscosity transfers itself throughout the entire fluid mass, beginning with the boundary, as soon as the state of rest ceases." Boussinesq postulated that "at a given time, a constant force, k, parallel to the horizontal axis is exerted on a unit mass throughout the fluid, in a manner to set the fluid in motion, except that the bottom maintains its own adherence to the boundary."

We therefore take $\dot{x}^i$ as a set of rectangular cartesian coordinates and consider an infinite fluid occupying all space in the upper half plane, $x \geq 0$, and take the z-axis vertical. Then, for

$$t < 0; \quad f^i = 0, \; \dot{x}^i = 0$$

for

$$t \geq 0; \quad f^i = f(i) = (k, 0, 0)$$
$$\dot{x}^i = x(i) = (u, 0, 0)$$
$$u = g(z, t).$$

The boundary conditions demand that

$$g(0, t) = 0, \quad g(z, 0) = 0.$$

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[1] M.J. Boussinesq, "On the manner in which viscosity enters into the problem in a fluid which goes into motion from a state of rest and on its effects toward preventing the existence of a velocity potential," Académie des Sciences Comptes Rendus, XC, (1880), 736-739; 967-969.

The inertial terms are given by

$$\ddot{X}(i) = \left(\frac{\partial w}{\partial t}, 0, 0\right)$$

The rate of deformation tensor is

$$d(ij) = \begin{bmatrix} 0 & 0 & \frac{1}{2}\frac{\partial w}{\partial z} \\ 0 & 0 & 0 \\ \frac{1}{2}\frac{\partial w}{\partial z} & 0 & 0 \end{bmatrix}$$

and the extra stress tensor is

$$t(ij) = (-p + \mathcal{T}_0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \mu \frac{\partial w}{\partial z}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$+ K_0\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \frac{1}{2}\frac{\partial w}{\partial z}K_1\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \frac{1}{4}\left(\frac{\partial w}{\partial z}\right)^2 K_2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with

$$\mathrm{I} = \mathrm{III} = 0\,, \quad \mathrm{II} = -\frac{1}{4}\left(\frac{\partial w}{\partial z}\right)^2$$

$$K_r = f(\mathrm{II}).$$

The Characteristic Kelvin and Poynting effects of the non-Newtonian fluid theory are observed. If the additional normal stress be wanting in the vertical direction, the boundary will tend to move up or down according to the sign of K_2. The physical effect of the additional normal stress in the direction of fluid motion is not as easily interpreted. We shall discuss this further below.

Equations of motion. - Recalling that $\mathcal{T}_r\left(\frac{\partial w}{\partial z}\right) = f(z,t)$, the equations of motion take the form

$$\rho \frac{\partial u}{\partial t} - \rho k = -\frac{\partial p}{\partial x} + \frac{1}{2}\frac{\partial}{\partial z}\left(\mathcal{F}_1 \frac{\partial u}{\partial z}\right)$$

$$0 = -\frac{\partial p}{\partial y} \tag{9-1}$$

$$0 = \frac{\partial}{\partial z}\left[-p + \mathcal{F}_0 + \frac{1}{4}\mathcal{F}_2\left(\frac{\partial u}{\partial z}\right)^2\right].$$

Solutions by Cases.

Case I. - Classical case. Suppose $\mathcal{F}_0 = \mathcal{F}_2 = 0,\ \mathcal{F}_1 = 2\mu$ and ρ is constant. Then (9-1) reduces to

$$\rho \frac{\partial u}{\partial t} - \rho k = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2}$$

$$0 = \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z}. \tag{9-2}$$

Now from $(9\text{-}2)_2$, p is a function of x only. From $(9\text{-}1)_1$

$$p = a_1 x + a_2$$

where a_1 and a_2 are constants. Boussinesq[2] studied the case where $a_1 = 0$, and gave, as a solution to the flow equation,

$$u = kt\left[1 - \frac{2}{\sqrt{\pi}}\int_{\omega_0}^{\infty}\left(1 - \frac{\omega_0^2}{\omega^2}\right)e^{-\omega^2}\,d\omega\right] \tag{9-3}$$

where

$$\omega_0 = \frac{z}{2}\sqrt{\frac{\rho}{\mu t}}.$$

Case IIA. - The relationship between assumptions concerning the pressure and existence of flows of non-Newtonian fluids, as discussed previously, is demonstrated in the problem of Boussinesq when $\mathcal{F}_1$ is considered constant, in which case (9-1) becomes

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[1]Ibid.

$$\rho\frac{\partial u}{\partial t}-\rho k=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial z^2}$$

$$0=-\frac{\partial p}{\partial y} \tag{9-4}$$

$$0=\frac{\partial}{\partial z}\left[-p+\frac{1}{4}\tau_2\left(\frac{\partial u}{\partial z}\right)^2\right]$$

where, from $(9\text{-}4)_2$

$$p=\frac{1}{4}\tau_2\left(\frac{\partial u}{\partial z}\right)^2+A_1(x,t). \tag{9-5}$$

Supposing, now, that we assume that p is a constant. Then from the explicit relationship for p, since $\left(\frac{\partial u}{\partial z}\right)^2$ is a function of z and t only, it must be the case that

$$\frac{\partial u}{\partial z}=A_2(t) \quad \text{and} \quad A_1=A_1(t)$$

hence

$$u=zA_2(t)+A_3(t) \tag{9-6}$$

that is, the velocity is a linear function of the distance from the boundary and therefore the second derivative vanishes. The right hand side of $(9\text{-}4)_1$ therefore vanishes, and

$$\frac{\partial u}{\partial t}=k$$

whence

$$u=kt+A_4(z). \tag{9-7}$$

A comparison of (9-6) with the above yields

$$A_3(t)=kt$$

$$A_2(t)=const.\equiv K$$

$$A_4(z)=Kz$$

so that the velocity is of the form

$$u(z,t) = kt + Kz. \tag{9-7}$$

The boundary conditions, applied to this form, restrict the velocity field to the trivial case, u identically zero.

We have therefore shown that, under assumption of constant pressure and f_r constant, there can be no Boussinesq flow of a non-Newtonian fluid.

Viguier[3] discussed the problem of Boussinesq in the case where the velocity gradients are not negligible, employing, in his equations of motion, viscosity forces as odd functions of the velocity gradients as earlier introduced[4] by himself.

Equation (9-7) comes about as a direct consequence of assuming that f_r and p are constant. Since $u(z,0) = 0$, it must be the case, as noted from the conclusion above regarding the impossibility of the flow, that $u(z,t) = kt$. Now the assumption of adherence to the boundary gives $u(0,t) = 0$ from which must be concluded that the velocity is identically zero. I have interpreted this result as indicating that the assumption of the constancy of pressure should not be made and remove this condition in the cases studied below.

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[3] G. Viguier, "Some thoughts on a problem of M.J. Boussinesq," Académie royale de Belgique Bulletion Classe des sciences, 1950-1, 71-76.

[4] G. Viguier, "The equation of the boundary layer in the case of higher order velocity gradients," Académie des Sciences Comptes Rendus, CCXXIV, (1947), 713.

so that the velocity is of the form

$$[illegible] \qquad (3.17)$$

The boundary condition, applied in this form, requires the velocity [illegible] to the [illegible] to be identically zero.

We thus find that, under the assumption of constant pressure and μ constant, there can be no homogeneous flow of a non-Newtonian fluid.

Rivlin[5] discussed the problem of homogeneous [illegible] in the case where the velocity gradients are not negligible, applying, in the equations of motion, viscosity forces as [illegible] functions of the velocity gradients he earlier introduced by himself[6].

Equation (3.17) comes about as a direct consequence of assuming that μ and p are constant. Since [illegible], it must be noted that the conclusion arises [illegible] the impossibility of [illegible]. But the assumption of adherence to the boundary gives [illegible] from which must be concluded that the velocity is identically zero. I have interpreted this result as indicating that the assumption of the constancy of pressure should not be made and remove this condition in the cases studied below.

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[5] R. Rivlin, "Some thoughts on a problem of [illegible]," [illegible]

[6] R. Rivlin, "The equations of the [illegible] in the case of higher order velocity gradients," [illegible]

Viguier retains the assumption of constancy of pressure and removes the condition of adherence, stating "the condition of adherence to the boundary does not seem to be satisfied. However one may attempt to give partial justification."

Case IIB. - If we assume that the pressure is a function of the flow direction and time only, similar considerations yield precisely the same results as in Case IIA above.

Case IIIA. - We are inevitably led in any study of the problem of Boussinesq, from the point of view of non-Newtonian fluids, to assumption of the dependence of the pressure distribution on the distance from the boundary. We consider $\mathcal{F}_1$, and ρ constant and assume $p = p(z,t)$.

Then from (9-1), we obtain,

$$\frac{\partial u}{\partial t} - k = \frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2} \tag{9-8}$$

$$p = \frac{1}{4}\left(\frac{\partial u}{\partial z}\right)^2 \mathcal{F}_2 + \phi(t)$$

We may take, as a solution for the first equation, that given by Boussinesq

$$u = kt\left[1 - \frac{2}{\sqrt{\pi}}\int_{\omega_o}^{\infty}\left(1 - \frac{\omega_o^2}{\omega^2}\right)e^{-\omega^2}d\omega\right] \tag{9-9}$$

where, as before

$$\omega_o = \frac{1}{2}z\left(\frac{\mu}{\rho}t\right)^{-1/2}$$

Differentiating (9-9) with respect to z yields

$$\frac{\partial u}{\partial z} = -\frac{k\rho}{\mu\sqrt{\pi}}\int_{\omega=\omega_o}^{\infty}e^{-\omega^2}d\left(\frac{1}{\omega}\right)$$

which can be rewritten, after integration by parts,

$$\frac{\partial w}{\partial z} = 2k\sqrt{\frac{t\ell}{\pi u}}\left(e^{-\omega_0^2} - 2\omega_0\int_{\omega_0}^{\infty} e^{-\omega^2}d\omega\right)$$

from which the pressure may be displayed,

$$p(z,t) = \frac{7_2 k^2 t\ell}{\pi\mu}\left[e^{-\omega_0^2} - 2\omega_0\int_{\omega_0}^{\infty} e^{-\omega^2}d\omega\right]^2 + A_2(t). \quad (9\text{-}1($$

We consider the growth properties of the pressure. It is clear that the pressure is constant downstream. At the boundary,

$$p(0,t) = \frac{7_2 k^2 t\ell}{\pi\mu} + A_2(t).$$

Returning to (9-10),

$$e^{-\omega_0^2(z,0)} = 0$$

and the other term in the square bracket is easily seen to vanish as $t \to 0$. Hence $p(z,0)$ approaches $A_2(0)$ as one would expect. An additional initial condition is necessary to establish the function $A_2(t)$ uniquely.

Now, as Z increases without bound

$$\omega_0\int_{\omega_0}^{\infty} e^{-\omega^2}d\omega \to 0$$

and therefore $p(z,t)$ approaches $\phi(t)$.

Finally, as t increases, since $\int_0^{\infty} e^{-\omega^2}d\omega = \frac{\sqrt{\pi}}{2}$

$$p(z,t) \longrightarrow \frac{7_2 k^2 t\ell}{\pi\mu} + \phi(t).$$

<u>Case IIIB</u>. - Supposing now that the pressure is a function of X, Z, and t. Then

$$\rho \frac{\partial u}{\partial t} - \rho k = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2}$$

and $$p = \frac{1}{4}\gamma_2 \left(\frac{\partial u}{\partial z}\right)^2 + \phi(x,t). \qquad (9\text{-}11)$$

From $(9\text{-}11)_1$, since u is independent of x, $\frac{\partial p}{\partial x}$ cannot be a function of x. Therefore, from $(9\text{-}11)_2$,

$$\frac{\partial p}{\partial x} = \frac{\partial \phi}{\partial x} = A_1(t)$$

and therefore

$$\phi(x,t) = x A_1(t) + A_2(t).$$

Hence we obtain, from (9-11),

$$\frac{\partial u}{\partial t} + \frac{1}{\rho} A_1(t) = k + \frac{\mu}{\rho} \frac{\partial^2 u}{\partial z^2}$$

with the integral

$$u = kt\left[1 - \frac{2}{\pi}\int_{\omega_0}^{\infty}\left(1 - \frac{\omega_0^2}{\omega}\right) e^{-\omega^2} d\omega\right] - \frac{1}{\rho}\int^{t} A_1(t)\,dt.$$

Now since $u(0,t)=0$, and the term in the bracket vanishes at $z=0$, it must be the case that $\int^t A_1 dt$ vanishes identically and therefore A_1 vanishes identically.

We point out, finally that in the completely general case when $\gamma_x = f\left(\frac{1}{4}\left(\frac{\partial u}{\partial z}\right)^2\right)$, from consideration of the general explicit form of the pressure, we see once more that the pressure must be a function of the distance from the boundary.

In conclusion, we have demonstrated that the second order theory of the flow of an incompressible non-Newtonian fluid demands, in the problem of Boussinesq, that the

pressure distribution must be of specific functional dependence. It cannot be considered constant without forcing the classical case or a trivial case and, further, the pressure cannot be a function of the flow direction.

X. THE DRAINING OF A VERTICAL PLATE

General. - We solve the equations of flow of a flat plate of infinite width which is drawn from an infinite fluid and compute the thickness of the fluid film on the surfaces of the plate. We assume a condition of steady state has been reached, that the fluid is incompressible and that the velocity is vertical and independent of the height.

This problem was first considered by Goucher and Ward[1] and an equivalent problem studied by Jeffreys[2].

We consider a rectangular cartesian coordinate system with the y-axis horizontal and perpendicular to the vertical plate; the z-axis is vertical and parallel to the plate, so that when $y = T$, the thickness of the film, $w = 0$ and $w' = 0$[3]; and that $w = V$, a constant (i.e., the vertical velocity of the plate), when $y = 0$, w being a function of y only. We disregard the curvature of the surface in the y-z plane and assume the acceleration of the fluid is negligible.

Under the above assumptions, the velocity vector is given by,

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[1] F.S. Goucher and H. Ward," A problem in viscosity: the thickness of liquid films on solid surfaces under dynamic conditions," Philosophical Magazine and Journal of Science, XLIV, (1922), 259-264 & 1002-1014.

[2] H. Jeffreys, "The draining of a vertical plate," Proceedings of the Cambridge Philosophical Society, XXVI, (1930), 204-205

[3] This is equivalent to assuming that there are no normal stresses on the free surface.

I. DRAINAGE OF A VERTICAL PLATE

Summary. - We study the drainage of fluid on a flat plate vertically withdrawn from an infinite fluid and determine the thickness of the fluid film on the surface of the plate. We assume a condition of steady state has been reached, that the fluid is incompressible and that the velocity is vertical and independent of the height.

This problem was first considered by Landau and Levich[1] and in a different context by Jeffreys[2].

We consider a rectangular cartesian coordinate system, with the y-axis horizontal and perpendicular to the vertical plate and the z-axis vertical and parallel to the plate, so that when $y = T$, the thickness of the film, $w \neq 0$ and $\partial w/\partial y = 0$[3], and that $w = V$, a constant (i.e., the vertical velocity of the plate), when $y = 0$, and which is a function of y only. We designate the coordinate of the surface in the y-z plane and assume the acceleration of the fluid is negligible.

Under the above assumptions, the velocity vector is given by:

[1] L. D. Landau and V. Levich, "A problem of viscosity: the withdrawal of liquid films on solid surfaces under dynamic conditions," Acta Physicochimica URSS, 17 (1942), 42-54.

[2] H. Jeffreys, "The draining of a vertical plate," Proceedings of the Cambridge Philosophical Society, 26 (1930), 204-205.

[3] This is equivalent to assuming that there are no normal stresses on the free surface.

$$\dot{x}(i) = (0, 0, w(y)).$$

The boundary conditions are:

$$w(T) = 0$$
$$w'(T) = 0$$
$$w(0) = V$$

the external force field is:

$$f(i) = (0, a, -g)$$

and the rate of deformation tensor is

$$d(ij) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}w' \\ 0 & \frac{1}{2}w' & 0 \end{bmatrix}.$$

The stress tensor takes the form

$$t(ij) = -p\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \mu w'\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$+ \frac{1}{2}w'K_1\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \frac{1}{4}w'^2K_2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where

$$I = III = 0, \quad II = -\frac{1}{4}w'^2 \qquad K_r = f(II).$$

and the Poynting effect is noted. An additional stress along the y-axis is noted; if this stress be wanting, the thickness of the film will tend to increase.

Equations of motion. - The equations of motion take the form

$$0 = \frac{\partial p}{\partial x}$$

$$0 = \frac{\partial}{\partial y}\left(-p + \frac{1}{4} \mathcal{F}_2 w'^2\right) \qquad (10\text{-}1)$$

$$\rho g = \frac{\partial}{\partial y}\left(\frac{1}{2} \mathcal{F}_1 w'\right) - \frac{\partial p}{\partial z}.$$

From $(10\text{-}1)_2$, if p is constant, then $\mathcal{F}_2 w'^2$ is constant. But $\mathcal{F}_2 w'^2$ is a power series in w'^2, and therefore it must be the case that w'^2 is constant. Since w' vanishes when $y = T$, w' must be identically zero, hence w must be constant. Since $w = 0$ when $y = T$, w must also vanish identically and the flow is trivial. We have demonstrated that, in the draining of a plate being removed from a non-Newtonian fluid, the pressure cannot be assumed constant, as in the classical case.

The solution. - Since the pressure is independent of x, we have from $(10\text{-}1)_2$

$$p = \frac{1}{4} \mathcal{F}_2 w'^2 + A_3(z). \qquad (10\text{-}2)$$

Integrating $(10\text{-}1)_3$ with respect to z, we obtain

$$p = z\left[\frac{\partial}{\partial y}\left(\frac{1}{2} \mathcal{F}_1 w'\right) - \rho g\right] + A_4(y). \qquad (10\text{-}3)$$

Comparing (10-2) and (10-3), we may write

$$p = (A_1 - \rho g) z + \frac{1}{4} \mathcal{F}_2 w'^2 + \text{const.}$$

and

$$\frac{\partial}{\partial y}\left(\frac{1}{2} \mathcal{F}_1 w'\right) = A_1 \equiv \rho g,$$

the choice of A_1 following from the absence of normal stresses.

By integration and, from the boundary condition for w', we obtain

$$\frac{1}{2}\eta_1 w' = A_1(y-T). \tag{10-4}$$

As in chapter V, we revert (10-4), obtaining

$$w' = \frac{2A_1}{a_0AB}(y-T) + \frac{2a_1AB^3A_1^3}{(a_0AB)^4}(y-T)^3 + \cdots\cdots$$

which, upon integrating and substituting $(3-5)_2$,

$$w = \frac{A_1}{2\mu}(y-T)^2 + \frac{a_1AB^3A_1}{32\mu^4}(y-T)^4 + \cdots\cdots \tag{10-5}$$

When $y=0$

$$w = V = \frac{A_1T^2}{2\mu} + \frac{a_1AB^3A_1^3}{32\mu^4}T^4 + \cdots\cdots\cdots \tag{10-6}$$

We may therefore combine (10-5) and (10-6) and write

$$w(y) - V = \frac{A_1}{2\mu}(y^2 - 2yT) + \frac{a_1AB^3A_1^3}{32\mu^4}(y^4 - 4y^3T + 6y^2T^2 - 4yT^3) + \cdots\cdots$$

The first approximation,

$$w - V = \frac{A_1}{2\mu}(y^2 - 2yT) = \frac{\rho g}{2\mu}(y^2 - 2yT)$$

is the solution of Goucher and Ward.

We now seek numerical values for the thickness, T, in the second approximation, in order to compare these values with those reported experimentally by Goucher and Ward. To this end, we take only the first two terms on the right in the expansion given by (10-6), and may therefore write

$$\frac{\Phi A_1^3}{32\mu^4}T^4 + \frac{A_1}{2\mu}T^2 - V = 0 \qquad (10\text{-}7)$$

where $\Phi \equiv a_1 A B^3$ and is a constant dependent only on the fluid much the same as μ. We see that $a_1 = 0$ implies that $\Phi = 0$ and

$$T^2 = \frac{2\mu V}{A_1}$$

<u>The thickness in the second approximation</u>. - From (10-7), we write the expression for the square of the thickness

$$T^2 = \frac{8\mu^3}{\Phi A_1^2}\left[-1 \pm \sqrt{1 + \frac{\Phi A_1 V}{2\mu^2}}\right] \qquad (10\text{-}8)$$

In order to compare the experimentally measured thicknesses with those given in (10-8), we take

$$\mu = .25\,, \quad A_1 = 1420$$

in c.g.s. units, which fall within the range of the measurements of Goucher and Ward.

Using a point from the experimental results, we then compute an value for Φ in c.g.s. units (dimension $\frac{MT}{L}$). Φ is negative[3] and in the interval

$$-10 \leqslant \Phi \times 10^6 \leqslant -1$$

with a value of -8.6×10^{-6} yielding a curve which compares

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[3] If $\tau_1 = f(\frac{1}{4}\omega^2)$ instead of $f(-\frac{1}{4}\omega^2)$, Φ will be positive.

very well with observation. We rewrite $-\Phi$ for Φ in (10-8)

$$T^2 = \frac{8\mu^2}{\Phi A_1^2}\left[1 \pm \sqrt{1 - \frac{\Phi A_1 V}{2\mu^2}}\right]. \tag{10-9}$$

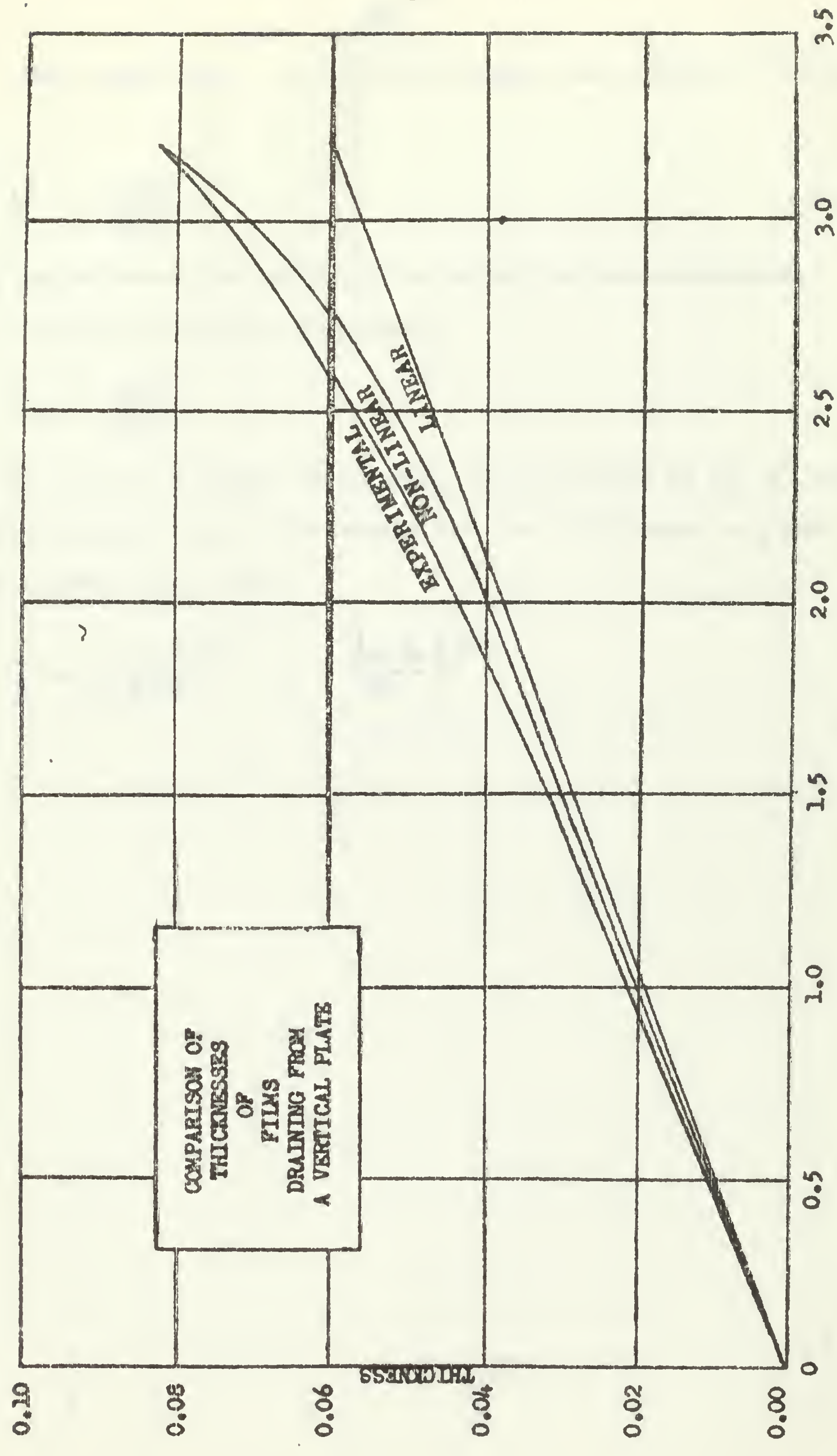
From knowledge of the dependence of the thickness, T, on the velocity, V, in the classical case, and from the available experimental results, we know that the thickness increases with an increase in the velocity. Therefore the expression in (10-9) with the negative sign is that which yields the solution to the physical problem.

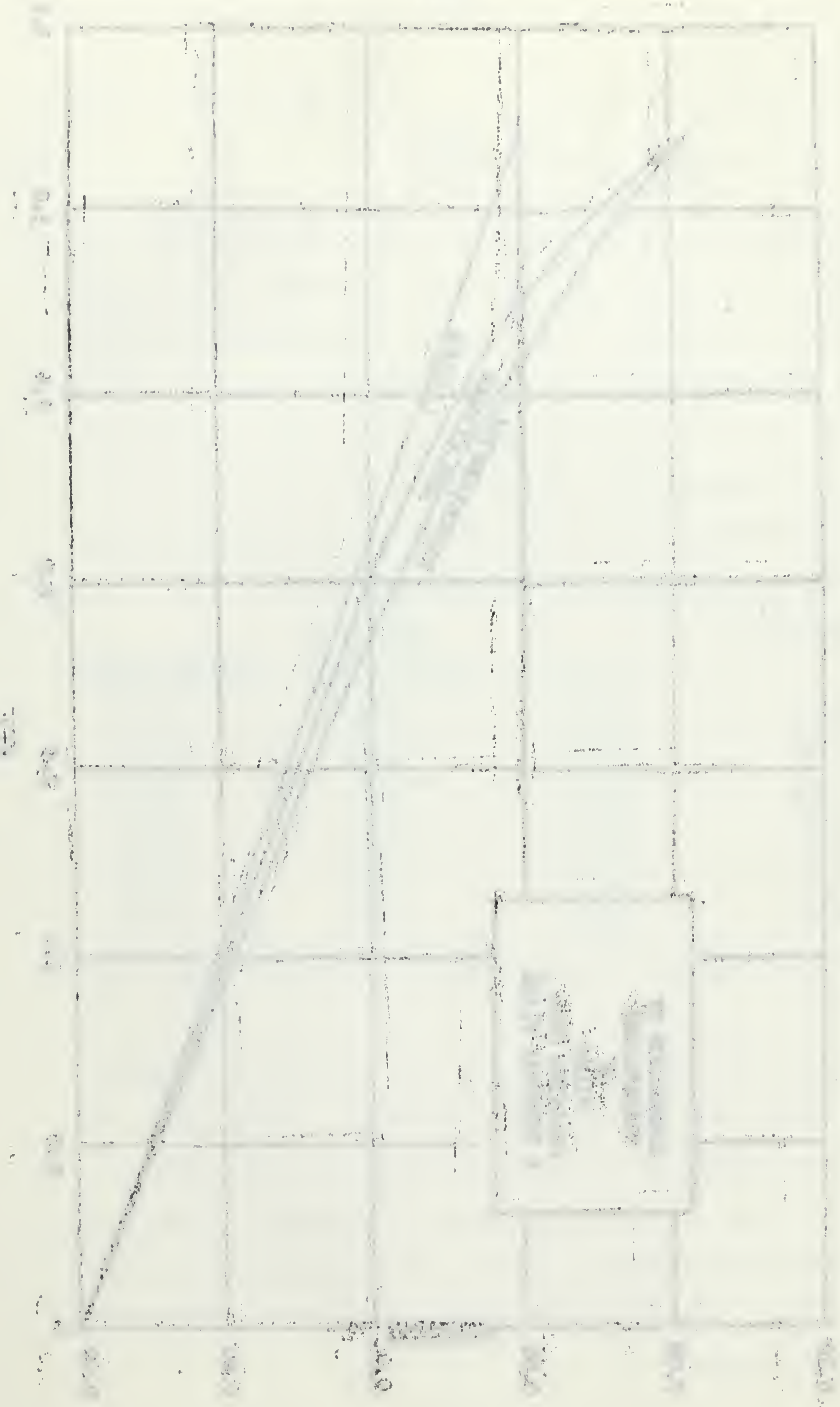
The following table shows comparative values in the classical, non-linear and experimental cases:

VELOCITY	THICKNESS		
	linear	non-linear	experimental
0.00	.0000	.0000	.0000
0.16	.0075	.0076	.0078
0.64	.0150	.0151	.0165
1.44	.0225	.0229	.0255
2.56	.0300	.0311	.0345
4.00	.0375	.0398	.0438
5.76	.0450	.0494	.0546
7.84	.0525	.0610	.0669
10.24	.0600	.0820	.0822

The above values of T are plotted as a function of $\sqrt{V}$ in figure 2.

Now, in the classical case, the thickness increases without bound, as the velocity increases. Intuitively, we would expect that, for a given fluid, i.e., for a given Φ, ρ, and μ, there probably is a velocity beyond which the assumptions regarding laminar flow and boundary conditions at $y = T$ are no longer valid. This is evident in the non-linear theory.


COMPARISON OF THICKNESSES OF FILMS DRAINING FROM A VERTICAL PLATE
EXPERIMENTAL
NON-LINEAR
LINEAR
THICKNESS
0.10
0.08
0.06
0.04
0.02
0.00
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
$\sqrt{V}$



The expression in (10-9) no longer has physical meaning when

$$V > \frac{2\mu^2}{\mathcal{S} A_1}. \tag{10-10}$$

At the point where the velocity is equal to the expression on the right in (10-10), we obtain

$$V_m = \frac{2\mu^2}{\mathcal{S} A_1}$$

where V_m is the maximum value that can be taken on by V, so that the maximum value attainable for the thickness is given by for a particular fluid

$$T = \left(\frac{8\mu^3}{\mathcal{S} A_1^2}\right)^{1/2} = \left(\frac{4\mu V_m}{A_1}\right)^{1/2}.$$

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APPENDIX

APPENDIX

DERIVATION OF THE EQUATIONS OF HYDRODYNAMICS[1]

1. Equation of continuity. - The equation of continuity expresses the conservation of mass. Suppose that a sufficiently smooth surface, S , includes a volume of fluid mass, V . We define the mass, m ,

$$m \equiv \int_V \rho \, dV$$

where $\int_V$ indicates integration over the volume V ; ρ is the fluid density and dV an element of volume. We wish to express the principle that mass is neither created nor destroyed. Hence the time rate of flow of mass from a given stationary volume must vanish, except for any mass which is added or subtracted from the volume.

We let $\frac{\partial m}{\partial t}$ be the rate of increase of mass. The local mass flux across an element of area (directed outwardly) dS_i per unit time is given by $\rho \dot{x}^i dS_i$ where $\dot{x}^i$ is the velocity vector. Therefore to express conservation of mass, we write,

$$\frac{\partial m}{\partial t} = -\oint_S \rho \dot{x}^i dS_i$$

where $\oint_S$ indicates integration over the surface S . Hence

$$\frac{\partial}{\partial t} \int_V \rho \, dV + \oint_S \rho v^i dS_i = 0$$

Assuming ρv^i and ρ continuously differentiable, an applying Green's Theorem, we obtain

$$\int_V \left(\frac{\partial \rho}{\partial t} + (\rho \dot{x}^i)_{,i} \right) dV = 0$$

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[1]Following the manner of Truesdell in lectures on Hydrodynamics at Indiana University, 1st semester, 1951-1952.

whence we obtain the Eulerian Equation of Continuity,

$$\frac{\partial \rho}{\partial t} + (\rho \dot{x}^i)_{,i} = 0$$

2. Cauchy's Laws of Motion (1821, 1828) From an axiomatic approach to continuum mechanics, we have that the force and moment acting on a body consists of three parts:

i. A force and moment equal to resultant force and moment, respectively, of a certain vector field, $t^i_{(n)}$, defined on the boundary, S, of a body; thus

$$\mathcal{F}^i = \oint_S t^i_{(n)}\, dS \qquad \mathcal{L}_i = \oint_S \varepsilon_{ijk} x^j t^k_{(n)}\, dS$$

where $\varepsilon_{ijk} = 1$ if i, j, k are an even permutation of $1, 2, 3$ or -1 if an odd permutation or 0 if i, j, k are not all different. We call $t^i_{(n)}$ the stress vector and shall see that dimension $t^i_{(n)}$ is $\frac{M}{L^2 T^2}$.

ii. A force and moment equal to the resultant force and momentum respectively of a certain density,

$$f^i_{PQ}\,(x^i_P - x^i_Q,\ \dot{x}^i_P - \dot{x}^i_Q,\ t)$$

where P, Q are two points. Then

$$\mathcal{F}^i = \int_{\text{body}} f^i_m\, dm \qquad \mathcal{L}^i = \int_{\text{body}} \varepsilon_{ijk} x^j f^k_m\, dm$$

where $f^i_m \equiv \int f^i_{PQ}\, dm(Q)$ over the whole space represents the effect of f^i_{PQ} on a point P as Q passes through all points of the space.

iii. A force and moment equal to the resultant force

and moment respectively of a certain moment density $f_E^i(x^i_{,u}, \dot{x}^i, t)$ plus a certain moment density $l_{E_i}(x^i, \dot{x}^i, t)$; then

$$\mathcal{F}^i = \int_{body} f_E^i \, dm \qquad \mathcal{L}_i = \int_{body} (\varepsilon_{ijk} x^j f_E^k + l_{E_i}) \, dm$$

and $f^i = f^i_m + f^i_E$ is the extraneous force field.

We therefore have the axioms of continuum mechanics

$$\dot{\mathcal{P}}^i = \oint_S t^i_{(n)} \, dS + \int_{body} f^i \, dm = \ddot{x}^i \, dm$$

$$\dot{\mathcal{H}}_i = \oint_S \varepsilon_{ijk} x^j t^k_{(n)} \, dS + \int_{body} (\varepsilon_{ijk} x^j f^k + l_i) \, dm = \int_0 \varepsilon_{ijk} \dot{x}^j \ddot{x}^k$$

where $\mathcal{P}^i$ and $\mathcal{H}_i$ are the linear and angular momentum respectively. The subscript (n) refers to the side of a particular surface, S , on which $t^i_{(n)}$ acts.

3. Cauchy's Laws of Motion.

i. Cauchy's First Theorem Let S be any surface (assumed regular enough to allow application of Stokes' and Green's Theorems), $t^i_{(n+)}$ and $t^i_{(n-)}$ be the vector densities on the $+$ side and $-$ side of the surface, S , each continuous on its own side; then if $f^i - \ddot{x}^i$ is of bounded variation, $t^i_{(n+)} = t^i_{(n-)}$

Proof: This theorem is proved from the equation for $\dot{\mathcal{P}}^i$ applied to a surface on each side of S . Taking as a bound on the integral of $f^i - \ddot{x}^i$, and allowing the surfaces to shrink to S , we obtain

$$\int_{S^-} t^i_{(n-)} \, dS + \int_{S^+} t^i_{(n+)} \, dS = 0$$

whence from the assumed continuity,

$$t^i_{(n+)} = t^i_{(n-)}$$

ii. Cauchy's Second Theorem Using the expression for $\dot{p}^i$ we show the existence of a stress tensor, defined by

$$t^i_{(n)} = t^{ji} n_j$$

where n_j is the unit normal to a surface (directed outward). The use of the stress tensor allows us to describe the stresses on a surface without having to speak of the stress direction.

iii. Cauchy's Laws of Motion. - We consider only the continuously differentiable case. From the expression for $\dot{p}^i$,

$$\oint_S t^i_{(n)} \, dS + \int_V (f^i - \ddot{x}^i) \, dm = 0.$$

Now,

$$\oint_S t^i_{(n)} \, dS = \oint_S t^{ji} n_j \, dS = \oint_S t^{ji} \, dS_j = \int_V t^{ji}{}_{,j} \, dV$$

by Green's Theorem. Upon substituting $dm = \rho \, dV$, we obtain

$$\int_V \left[t^{ji}{}_{,j} + \rho(f^i - \ddot{x}^i) \right] dV = 0$$

or $t^{ji}{}_{,j} + \rho(f^i - \ddot{x}^i) = 0$

Cauchy's First Law of Motion.

From the expression for $\dot{H}^i$, we can similarily show that $\varepsilon_{ijk} t^{jk} + \rho l_i = 0$, Cauchy's Second Law of Motion.

Now if l_i , the extraneous moment of momentum vanishes, then $t^{ij} = t^{ji}$, which we henceforth assume.

4. The Energy Equatuion. - We define the kinetic energy, K ,

$$K \equiv \frac{1}{2} \int_V \rho \dot{x}^i \dot{x}_i \, dV.$$

We further consider the internal energy E of a body, and assume that it is a continuous function of mass. We may then consider an integral form $E=\int \varepsilon\, dm$ where ε is the internal energy density. E is changed by two specific processes, (1) mechanical work and (2) non-mechanical causes (e.g. radiation, heat conduction, etc.)

Principle of Energy. - Let there be a heat flux, q^i, (analogue of the stress vector) such that we may write

$$\dot{K}+\dot{E} = \int_V \rho f^i \dot{x}_i \, dV + \oint_S t^{ji} \dot{x}_i \, dS_j - \oint_S q^i \, dS_i$$

that is, the total time rate of change of energy is equal to the mechanical work performed plus the work done due to the stress tensor. $\oint q^i dS_i$ is so defined as to balance the equation. We note that the interconvertability of heat and work is already determined by this equation because of the fact that q^i is written in mechanical units, i.e.

$$\text{dimension} = \frac{\text{energy}}{\text{time area}} = \frac{M}{T^3}$$

Assuming all functions involved are continuously differentiable, substituting the expression for K and E and applying the divergence theorem, we obtain

$$\int_V \left[\dot{x}^i (\rho \ddot{x}_i - \rho f_i - t^j_{i,j})\right] dV + \int_V \left[\rho \dot{\varepsilon} - t^{ji} \dot{x}_{i,j} + q^i_{,i}\right] dV = 0$$

Applying Cauchy's first law, and from continuity assumptions, we have

$$\rho \dot{\varepsilon} = t^{ji} \dot{x}_{i,j} - q^i_{,i}.$$

Defining the rate of deformation tensor as usual,

$$d_{ij} \equiv \frac{1}{2}(\dot{x}_{i,j} + \dot{x}_{j,i})$$

and recalling the symmetry of the stress tensor, we have

$\rho\dot{\varepsilon} = t^{ij}d_{ij} - g^{i}{}_{,i}$ the Fourier-Kirchoff-C. Neumann equation. Thus $t^{ij}d_{ij}$, the stress power, is the rate at which stress does work per unit volume.

Assuming as usual for a homogeneous fluid that ε is given by $\varepsilon = \varepsilon(v,\eta)$, where $v = \frac{1}{\rho}$ and η is a local statistical parameter (entropy), we define $\pi = -\frac{\partial\varepsilon}{\partial v}$, $\theta = \frac{\partial\varepsilon}{\partial\eta}$, the pressure and temperature respectively; then $d\varepsilon = -\pi\, dv + \theta\, d\eta$.

We now define the extra stress tensor v^{i}_{j} by $v^{i}_{j} = p\delta^{i}_{j} + t^{i}_{j}$, where p is <u>any</u> scalar. After some algebraic manipulation, we obtain

$$\rho\dot{\varepsilon} = -p\, d^{k}_{k} + v^{ij}d_{ij} - g^{i}{}_{,i}$$

Hence

$$\rho\theta\dot{\eta} = \rho\pi\dot{v} - p\, d^{k}_{k} + v^{ij}d_{ij} - g^{i}{}_{,i}$$

From the equation of continuity

$$\operatorname{div} \dot{x}^{k} = d^{k}_{k} = \dot{v}$$

whence

$$\rho\theta\dot{\eta} = v^{ij}d_{ij} - g^{i}{}_{,i} + (\pi - p)\frac{\dot{v}}{v}$$

For an incompressible fluid $\dot{v} = 0$, and

$$\rho\theta\dot{\eta} = v^{ij}d_{ij} - g^{i}{}_{,i} .$$

VITA

The author, Commander Dominic Anthony Paolucci, United States Navy, was born on June 13, 1923 in Buffalo, New York, the son of Stefana and the late Christopher, Italian immigrants. He grew up in Buffalo, where he attended Hutchinson-Central High School from which he was graduated in June, 1940.

He was appointed to the United States Naval Academy by the late Honorable Pius L. Schwert, Member of Congress from the 42nd Congressional District of the State of New York. He entered the Naval Academy in July, 1940, was graduated in June, 1943 with the degree Bachelor of Science and was commissioned an Ensign in the Navy.

Commander Paolucci served on the Destroyer DOYLE and the Submarine RONCADOR during World War II; he served on the Submarines BECUNA and SEA DEVIL after the war.

In 1949, he entered the U.S. Navy Postgraduate School and studied there for one year in the Advanced Science curriculum prior to attending the Graduate Institute of Mathemathics and Mechanics at Indiana University from September, 1950 to June, 1952 under the sponsorship of the Office of Naval Research. He was awarded the degree Master of Arts (Mathematics) by Indiana University in September, 1951, and accepted as a candidate for the Ph.D degree in June 1952.

The author served as Executive Officer of the Submarine SEA LION from June, 1952 to July, 1954. Following this tour of sea duty, he served in the Office of Naval Research as Submarine Research Projects Officer until July, 1956.

From July, 1956 to July, 1957, the author commanded the Submarine BALAO and from July, 1957 to October, 1958, he commanded the GUPPY Submarine TRUTTA. A portion of this dissertation was written while he so served.

Commander Paolucci is now attached to the office of Admiral Arleigh Burke, USN, the Chief of Naval Operations, in the Pentagon, where he is the Head, Submarine Warfare Policy and Training Section.

The author is married to the former Thelma Jeanne Dailey of Greenville, Pennsylvania. They presently make their residence in Fairfax, Virginia with their three daughters, Emir, Carla and Kristin.

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